8 Let $F(x, y, t) = 0$ and $G(x, y, t) = 0$ be used to express x and y in terms of t. Find general formulas for dx/dt and dy/dt.

9 Let $z = f(xy)$. Show that this obeys the differential relation

$$x\left(\frac{\partial z}{\partial x}\right) - y\left(\frac{\partial z}{\partial y}\right) = 0$$

10 Let $w = F(xz, yz)$. Show that

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = z\frac{\partial w}{\partial z}$$

11 A function f is said to be homogeneous of degree k in a neighborhood $\mathcal{N}$ of the origin if $f(tx, ty) = t^k f(x, y)$ for all points $(x, y) \in \mathcal{N}$ and all t, $0 \le t \le 1$. Assuming appropriate continuity conditions, prove that f satisfies in $\mathcal{N}$ the differential equation

$$xf_1(x, y) + yf_2(x, y) = kf(x, y)$$

12 Setting $z = f(x, y)$, Exercise 11 shows that $x(\partial z/\partial x) + y(\partial z/\partial y) = 0$ whenever f is homogeneous of degree $k = 0$. Show that in polar coordinates this differential equation becomes simply $r(\partial z/\partial r) = 0$, and from this deduce that the general homogeneous function of degree 0 is of the form $f(x, y) = F(y/x)$.

13 If $z = F(ax + by)$, then $b(\partial z/\partial x) - a(\partial z/\partial y) = 0$.

14 If $u = F(x - ct) + G(x + ct)$, then

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

15 If $z = \phi(x, y)$ is a solution of $F(x + y + z, Ax + By) = 0$, show that $A(\partial z/\partial y) - B(\partial z/\partial x)$ is constant.

16 Show that the substitution $x = e^s$, $y = e^t$ converts the equation

$$x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) + x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 0$$

into the equation $\partial^2 u/\partial s^2 + \partial^2 u/\partial t^2 = 0$.

17 Show that the substitution $u = x^2 - y^2$, $v = 2xy$ converts the equation $\partial^2 W/\partial x^2 + \partial^2 W/\partial y^2 = 0$ into $\partial^2 W/\partial u^2 + \partial^2 W/\partial v^2 = 0$.

18 Show that if p and E are regarded as independent, the differential equation (3-32) takes the form

$$\frac{\partial T}{\partial p} - T\frac{\partial V}{\partial E} + p\frac{\partial(V, T)}{\partial(E, p)} = 0$$

19 Let f be of class C'' in the plane, and let S be a closed and bounded set such that $f_1(p) = 0$ and $f_2(p) = 0$ for all $p \in S$. Show that there is a constant M such that $|f(p) - f(q)| \le M|p - q|^2$ for all points p and q lying in S.

***20** (*Continuation of Exercise* 19) Show that if S is the set of points on an arc given by the equations $x = \phi(t)$, $y = \psi(t)$, where ϕ and ψ are of class C', then the function f is constant-valued on S.

21 Let f be a function of class C' with $f(1, 1) = 1$, $f_1(1, 1) = a$, and $f_2(1, 1) = b$. Let $\phi(x) = f(x, f(x, x))$. Find $\phi(1)$ and $\phi'(1)$.