$$g(x, y, z) = 2x + 3y + z - 12 = 0$$

for which $f(x, y, z) = 4x^2 + y^2 + z^2$ is minimum. We find that

$$z = 12 - 2x - 3y$$

so that

$$F(x, y) = 4x^2 + y^2 + (12 - 2x - 3y)^2$$

The critical points of F are found from the equations

$$0 = F_1(x, y) = 8x + 2(12 - 2x - 3y)(-2)$$

$$0 = F_2(x, y) = 2y + 2(12 - 2x - 3y)(-3).$$

These have only one solution, $(\frac{6}{11}, \frac{36}{11})$. Checking, we find that $F_{11} = 16$, $F_{12} = 12$, and $F_{22} = 20$, so that $\Delta = (12)^2 - (16)(20) < 0$ and this point yields a local minimum for F. Using the side condition to find z, we find that the solution to the original problem is the point $(\frac{6}{11}, \frac{36}{11}, \frac{12}{11})$.

A general approach to the solution of extremal problems with constraints, usually called the method of Lagrange multipliers, will be treated in Chap. 10. While of some theoretical interest, it has many practical limitations and is not often the most successful approach to a numerical solution.

EXERCISES

1 What are the maxima and minima of $f(x) = (2x^2 + 6x + 21)/(x^2 + 4x + 10)$?

2 (*a*) Does $P(x) = 1 - x + x^2/2 - x^3/3 + x^4/4$ have any real zeros?
(*b*) What about $Q(x) = P(x) - x^5/5 + x^6/6$?

3 Find the maximum and minimum value of $2x^2 - 3y^2 - 2x$ for $x^2 + y^2 \le 1$.

4 Find the maximum and minimum value of $2x^2 + y^2 + 2x$ for $x^2 + y^2 \le 1$.

5 Discuss the nature of the critical points of each of the functions described by:
(*a*) $f(x, y) = x^2 - y^2$ (*b*) $f(x, y) = 3xy - x^2 - y^2$
(*c*) $f(x, y) = 2x^4 + y^4 - x^2 - 2y^2$ (*d*) $f(x, y) = 4x^2 - 12xy + 9y^2$
(*e*) $f(x, y) = x^4 + y^4$ (*f*) $f(x, y) = x^4 - y^4$

6 Sketch the level curves of f for the functions given in Exercise 5, parts (*a*), (*b*), (*d*), and (*e*).

7 Given $f(x, y) = x^2 - 2xy + 3y^2 - x$ and the square $D = \{(x, y), 0 \le x \le 1, 0 \le y \le 1\}$. Find all critical points and find the maximum and minimum on D.

8 Show that $H(x, y) = x^2y^4 + x^4y^2 - 3x^2y^2 + 1 \ge 0$ for all (x, y).

9 Let $f(x, y) = (y - x^2)(y - 2x^2)$. Show that the origin is a critical point for f which is a saddle point, although on any line through the origin, f has a local minimum at $(0, 0)$.

10 Given n points in space, $P_1, P_2, \ldots, P_n$, find the point P for which

$$f(P) = \sum_1^n |P - P_j|^2$$

is a minimum.

11 Let $f \in C'$ in the open set Ω and have no critical points there. Let E be the set where $f(p) = 0$. Show that E has no interior points.

12 Find the point on the line through $(1, 0, 0)$ and $(0, 1, 0)$ which is closest to the line: $x = t$, $y = t$, $z = t$.

13 Find the maximum value of $x^2 + 12xy + 2y^2$, among the points (x, y) for which $4x^2 + y^2 = 25$.

14 Give a complete discussion of the problem of finding the right circular cone of greatest lateral area which may be inscribed upside down in the cone of radius 1 and altitude 3.

15 Given an equilateral triangular set, what location of P in the set will yield the maximum value of the product of the distances from P to the vertices? (*Hint:* Use the symmetry of the triangle.)

16 In the solution of the **normalized two-person game** whose **payoff matrix** is

$$\begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

one is led to the problem of finding the saddle points of the function F described by

$$F(x_1, x_2, x_3, y_1, y_2, y_3) = (x_1 - 2x_2 + x_3)y_1 + (2x_1 - 2x_3)y_2 + (-x_1 + x_2)y_3$$

subject to the constraints $x_1 + x_2 + x_3 = 1$, $y_1 + y_2 + y_3 = 1$. Show that the saddle point is $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $y = (\frac{2}{7}, \frac{1}{7}, \frac{4}{7})$.

***17** Let $(x_1, y_1) = P_1, (x_2, y_2) = P_2, \ldots, P_n$ be a set of points, not all the same. Find the line L which "fits" these points best in the sense that it minimizes $\sum_1^n d_j^2$, where d_j is the distance from P_j to L. [Note that this is not the same as the function f given in (3-46).]

18 Using Theorem 20, prove the following: If D is a closed bounded set, and if f and g are both harmonic in D, and if $f(p) = g(p)$ for all p on the boundary of D, then $f \equiv g$ in D.

19 For a function F of one variable, show that the method of steepest ascent leads to the algorithm

$$x_{n+1} = x_n + hF'(x_n)$$

If this is applied to find the maximum of $F(x) = x^4 - 6x^2 + 5$ for x in the interval $[-1, 1]$, show that the algorithm yields a convergent sequence $\{x_n\}$ if h is small, but that the method fails if h is too large and the starting point x_0 is unfortunately chosen.

20 Apply the method of steepest ascent to locate the maxima of the function $F(x, y) = x^3 - 3xy^2$ in the square

$$D = \{\text{all } (x, y) \text{ with } |x| \le 2, |y| \le 2\}$$

Examine the effect of the following three choices of initial point, $p_0 = (-1, 0), (1, 0), (0, 1)$, and the effect of the step size $h = .1$ and $h = .01$.

***21** Let $C_1 \ge C_2 \ge \cdots \ge C_n$ be a fixed set of positive numbers. Maximize the linear function $L(x_1, x_2, \ldots, x_n) = \sum_1^n C_j x_j$ in the closed set described by the inequalities $0 \le x_j \le 1$, $\sum_1^n x_j \le A$.