The general case follows the same pattern as Theorem 17. Given a system of m simultaneous equations

$$\begin{aligned} \phi_1(x_1, x_2, \ldots, x_n) &= 0 \\ \phi_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\cdots\cdots \\ \phi_m(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}$$

in n variables, and a point $p = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ that satisfies the system, we can (in theory) solve for a specific set of m of the variables, say $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$, in terms of the rest in a neighborhood of p if the Jacobian

$$\frac{\partial(\phi_1, \phi_2, \ldots, \phi_m)}{\partial(x_{i_1}, x_{i_2}, \ldots, x_{i_m})} \neq 0$$

at p.

EXERCISES

1 Can the curve whose equation is $x^2 + y + \sin(xy) = 0$ be described by an equation of the form $y = f(x)$ in a neighborhood of the point $(0, 0)$? Can it be described by an equation of the form $x = g(y)$?

2 Can the surface whose equation is $xy - z \log y + e^{xz} = 1$ be represented in the form $z = f(x, y)$ in a neighborhood of $(0, 1, 1)$? In the form $y = g(x, z)$?

3 The point $(1, -1, 2)$ lies on both of the surfaces described by the equations $x^2(y^2 + z^2) = 5$ and $(x - z)^2 + y^2 = 2$. Show that in a neighborhood of this point, the curve of intersection of the surfaces can be described by a pair of equations of the form $z = f(x)$, $y = g(x)$.

4 Study the corresponding question for the surfaces with equations $x^2 + y^2 = 4$ and $2x^2 + y^2 - 8z^2 = 8$ and the point $(2, 0, 0)$ which lies on both.

5 The pair of equations

$$\begin{cases} xy + 2yz = 3xz \\ xyz + x - y = 1 \end{cases}$$

is satisfied by the choice $x = y = z = 1$. Study the problem of solving (either in theory or in practice) this pair of equations for two of the unknowns as a function of the third, in the vicinity of the $(1, 1, 1)$ solution.

6 (a) Let f be a function of one variable for which $f(1) = 0$. What additional conditions on f will allow the equation

$$2f(xy) = f(x) + f(y)$$

to be solved for y in a neighborhood of $(1, 1)$?

(b) Obtain the explicit solution for the choice $f(t) = t^2 - 1$.

*7 With f again a function of one variable obeying $f(1) = 0$, discuss the problem of solving the equation $f(xy) = f(x) + f(y)$ for y near the point $(1, 1)$.

8 Using the method of Theorem 18, state and prove a theorem which gives sufficient conditions for the equations

$$F(x, y, z, t) = 0 \qquad G(x, y, z, t) = 0 \qquad \text{and} \quad H(x, y, z, t) = 0$$

to be solvable for x, y, and z as functions of t.

9 Apply Theorem 18 to decide if it is possible to solve the equations

$$xy^2 + xzu + yv^2 = 3 \qquad \text{and} \qquad u^3yz + 2xv - u^2v^2 = 2$$

for u and v as functions of (x, y, z) in a neighborhood of the points $(x, y, z) = (1, 1, 1)$, $(u, v) = (1, 1)$.

10 Find the conditions on the function F which allow you to solve the equation

$$F(F(x, y), y) = 0$$

for y as a function of x near $(0, 0)$. Assume $F(0, 0) = 0$.

11 Find conditions on the functions f and g which permit you to solve the equations

$$f(xy) + g(yz) = 0 \qquad \text{and} \qquad g(xy) + f(yz) = 0$$

for y and z as functions of x, near the point where $x = y = z = 1$; assume that $f(1) = g(1) = 0$.

7.7 FUNCTIONAL DEPENDENCE

In Sec. 7.5, we studied at some length the properties of transformations of class C' whose Jacobian is never 0 in an open set. We found that they map open sets onto open sets of the same dimension, are locally 1-to-1, and therefore have local inverses. In this section, we examine the behavior of a transformation T whose Jacobian vanishes everywhere in an open set.

We illustrate this first with a simple example. Consider the transformation described by

$$T: \begin{cases} u = \cos(x + y^2) \\ v = \sin(x + y^2) \end{cases}$$

At (x, y), the Jacobian of T is

$$\begin{aligned} J(x, y) &= \det \begin{bmatrix} -\sin(x + y^2) & -2y \sin(x + y^2) \\ \cos(x + y^2) & 2y \cos(x + y^2) \end{bmatrix} \\ &= -2y \sin(x + y^2) \cos(x + y^2) + 2y \sin(x + y^2) \cos(x + y^2) \\ &= 0 \end{aligned}$$

This transformation fails to have many of the properties which were shown to hold for those with nonvanishing Jacobian. For example, although it is continuous and in fact of class C^∞, it does not map open sets in the XY plane into open sets in the UV plane. Since $u^2 + v^2 = 1$ for any choice of (x, y), T maps the entire XY plane onto the set of points on this circle of radius 1. Furthermore, it is not locally 1-to-1. All the points on the parabola $x + y^2 = c$ map into the same point $(\cos c, \sin c)$, and as c changes, these parabolas cover the entire XY plane. Thus, any disk, no matter how small, contains points having the same image. Speaking on the intuitive level for the moment, T might be called a dimension-reducing transformation; if we regard open sets in the plane as two-dimensional, and curves as one-dimensional, then T takes a two-dimensional set into a one-dimensional set.