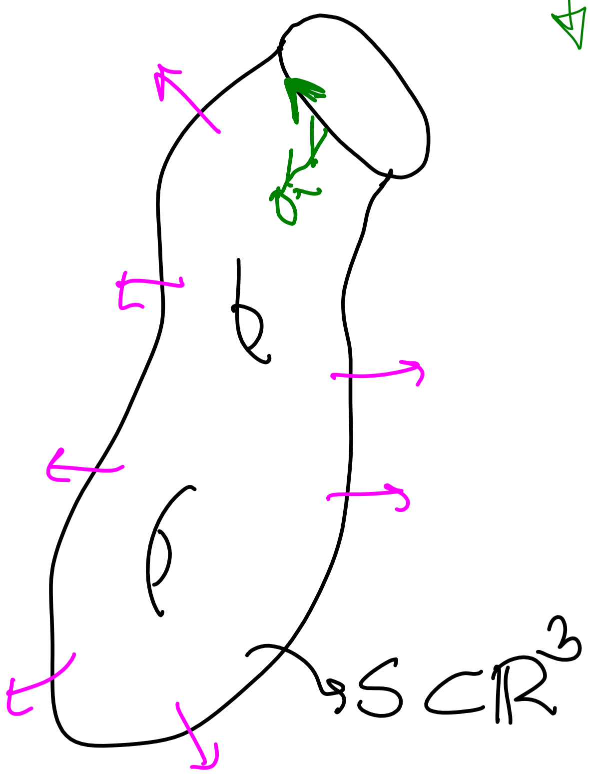
Stokes's Theorem

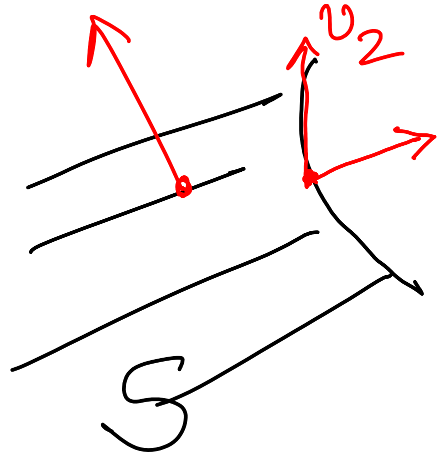
$$\int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dA$$

∂S: simple closed oriented curve — line integral

S oriented, C^1, parametrized over measurable W — surface integral

∂S is oriented as the boundary of surface S

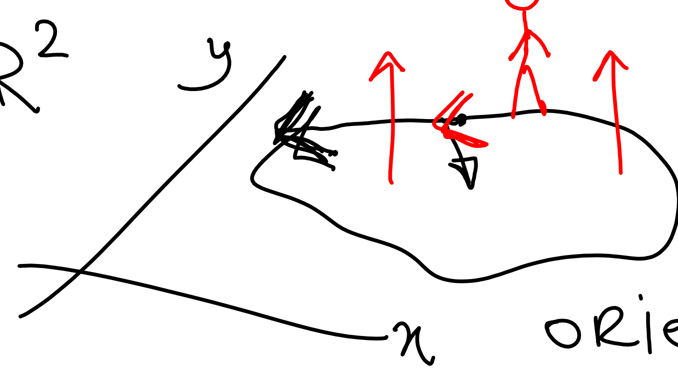




Choose v_2 s.t. $v_1 \times v_2 \parallel n$

v_1 ↳ directed outwards on ∂

corol. $W \subset \mathbb{R}^2$



orientation convention agrees with Green's.

Stokes's $\Rightarrow$ Green's

$\operatorname{curl} F = \operatorname{curl}(P, Q, 0) = (\partial_x Q - \partial_y P)\vec{k}$ & $\vec{n} = \vec{k}$

proof of Stokes's. $\vec{F} = (P, Q, R)$

$$\iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dA$$

$$= \iint_S \left(R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\right) \cdot \vec{n}\, dA$$

just for terms with P

$$= \iint_S \left(P_z \vec{j} - P_y \vec{k}\right) \cdot n\, dA$$

$$(x_u, y_u, z_u) = G_u \qquad G_v = (x_v, y_v, z_v)$$

$$= \iint_W \left(P_z \vec{j} - P_y \vec{k}\right) \cdot \left|G_u \times G_v\right| du\, dv$$

$$\left|G_u \times G_v\right| = \left(\ldots,\ x_v z_u - x_u z_v,\ x_u y_v - x_v y_u\right)$$

$$= \iint_W \left[P_z \cdot (x_v z_u - x_u z_v) - P_y \cdot (x_u y_v - x_v y_u)\right] du\, dv$$

$$x_v \cdot (P_z z_u + P_y y_u) = x_v \cdot P_u - \cancel{x_v P_x x_u}$$

$$\partial_u P = P_x x_u + P_y y_u + P_z z_u$$

$$-x_u (P_z z_v + P_y y_v) = -x_u P_v + \cancel{x_u P_x \cdot x_v}$$

$$= \iint_W \left(x_v P_u - x_u P_v \right) du\, dv$$

$$=$$

$$= \iint_W \Big(-\overbrace{\partial_v (P x_u)}^{= P_v x_u + \cancel{P x_{uv}}} + \overbrace{\partial_u (P x_v)}^{= P_u x_v + \cancel{P x_{uv}}} \Big)\, du\, dv$$

Green's

$$= \int_{\partial W} (P x_u, P x_v) \cdot d\vec{r}$$

$$= \int_{\partial W} P x_u\, du + P x_v\, dv \quad \text{in } \mathbb{R}^2 \supset W$$

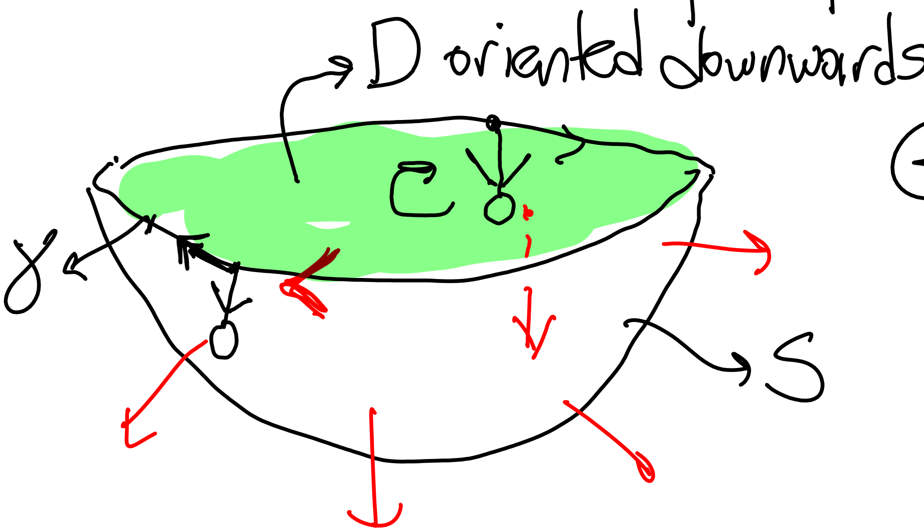
$$= \int_{\partial S} P\, dx$$

just the terms containing P

LHS: $$\int_{\partial S} \vec{F} \cdot \underbrace{d\vec{x}}_{= (dx, dy, dz)} = \int_{\partial S} P\, dx + Q\, dy + R\, dz$$

The proof for other terms is similar. $\blacksquare$

ex: S = lower half of $x^2+y^2+\frac{z^2}{16}=1$.



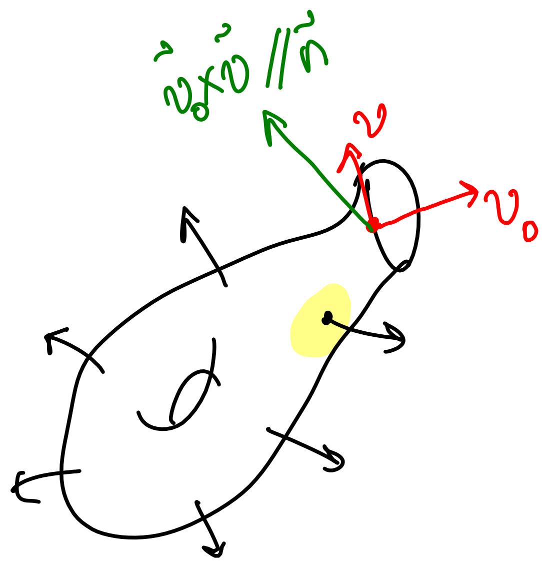
Given $\vec{F}:\mathbb{R}^3\to\mathbb{R}^3$ which is C^1.

unit normal vector field over S

Stokes's: $\iint_{S} \operatorname{curl}\vec{F}\cdot\vec{n}\,dA$ (oriented S)

$$= \int_{\gamma} \vec{F}\cdot d\vec{x}$$

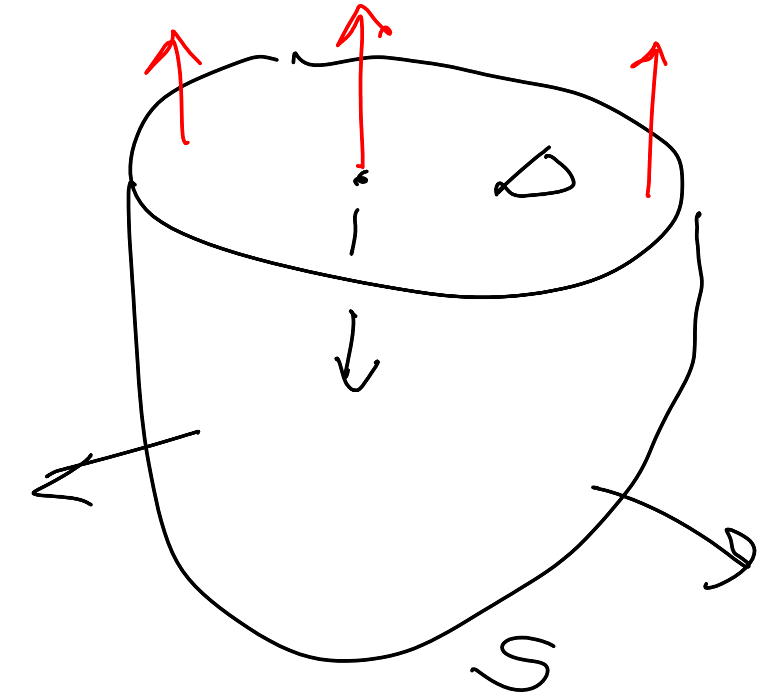
accordingly oriented $\partial S=\gamma$



$$= \iint_{D} \operatorname{curl}\vec{F}\cdot\vec{n}\,dA \quad (\text{oriented } \gamma=\partial D)$$

D oriented downwards

<u>Note</u>: γ oriented wrt S is the same as γ oriented wrt D.



Observe:

$$0 = \iint_{-D} \mathrm{curl}\,\vec{F}\cdot\vec{n}\,dA + \iint_{S} \mathrm{curl}\,\vec{F}\cdot n\,dA$$

Since $S-D$ is a closed surface we proved:

<u>thm</u>: The flux integral of the curl of any vector field over any **closed** srfc is 0.

<u>back to ex.</u> $\vec{F} = (2x, 2y, x^2+y^2)$

$$\iint_{S} \mathrm{curl}\,\vec{F}\cdot\vec{n}\,dA = \int_{\gamma} \vec{F}\cdot d\vec{x} = \iint_{D} \mathrm{curl}\,\vec{F}\cdot\vec{n}\,dA$$

S oriented downwards

$\gamma = \{x^2+y^2=1\}$

D oriented downwards $= \{x^2+y^2 \le 1\}$, $z=0$

The last integral vanishes because:

$$\begin{aligned} \operatorname{curl}\vec{F}\cdot\vec{n} &= \operatorname{curl}\vec{F}\cdot(-\vec{k}) \\ &= (\partial_x 2y - \partial_y 2x)\vec{k}\cdot(-\vec{k}) \\ &= 0 \end{aligned}$$

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2) The good big picture.

Every fnc below is C^1; domains are "nice".

(1) Divergence thm: $\iiint_T \operatorname{div}\vec{F}\,dV = \iint_{\partial T} \vec{F}\cdot\vec{n}\,dS$

Stokes' thm: $\iint_\Sigma \operatorname{curl}\vec{F}\cdot\vec{n}\,dS = \int_{\partial\Sigma}\vec{F}\cdot d\vec{x}$

Fund thm of line integrals: $\int_C \operatorname{grad} f\cdot d\vec{x} = f\Big|_{\partial C} = f(B)-f(A)$

C: from A to B

(2) If Σ is closed (i.e. $\partial\Sigma=\emptyset$) then $\iint_\Sigma \operatorname{curl}\vec{F}\cdot\vec{n}\,dS = 0$ always.

(3) If C is a closed curve (i.e. $\partial C = \emptyset$) then $\int_C \operatorname{grad} f \cdot d\vec{x} = 0$ always.

(4) Then when is a v.f. $\vec{F}$ gradient?

<u>thm</u>. The following are equivalent for $\vec{F}$:

(a) For fixed pts A, B, the line integral of $\vec{F}$ along any C from A to B is indep from the chosen C (path independence).

(b) Along any closed curve, the work done by $\vec{F}$ is 0. (conservative)

(c) There is a potential for $\vec{F}$, i.e. $\vec{F} = \operatorname{grad} f$ for some f.

<u>Proof</u>. (a) $\Leftrightarrow$ (b) easy. For the rest see Folland, Thm 5.60.

(5) If $\operatorname{curl} \vec{F} = 0$ then $\iint_\Sigma \operatorname{curl} \vec{F} \cdot \vec{n}\, dS = 0$

For example if $\vec{F} = \operatorname{grad} f$.

Q. Are there vector fields with $\text{curl}\, F = 0$ but they are not gradient vector fields?

A. Yes, E.g. $\vec{F} = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$.

The essential issue here is that $\text{dom}\,\vec{F} = \mathbb{R}^3 - \{z\text{-axis}\}$

⑥ thm. If $\Omega = \text{dom}\,\vec{F}$ is convex (more generally simply connected) then $\text{curl}\,\vec{F} = 0$ implies $\vec{F}$ is a gradient: it has a potential over Ω.

proof. See Folland thm 5.62.

⑦ Similarly $\text{div}\,\text{curl}\,\vec{G} = 0$ always.

8

Q. If $\operatorname{div} F = 0$, does that mean $F = \operatorname{curl}\vec{G}$ for some $\vec{G}$?

thm. If $\Omega = \operatorname{dom}\vec{F}$ is convex (more generally simply connected) then $\operatorname{div}\vec{F} = 0$ implies there is some $\vec{G}$ over Ω s.t. $\operatorname{curl}\vec{G} = \vec{F}$

proof. See Folland thm 5.63.

In general we Riem. integrate "differential forms". For a form ω, $d\omega$ is defined. Moreover always $dd\omega = 0$ (Like in 5 & 7).

We integrate over "smooth manifolds". The algebra of forms tells things about the domain (like in 6 & 8).

For further reading, start from Folland 5.9.