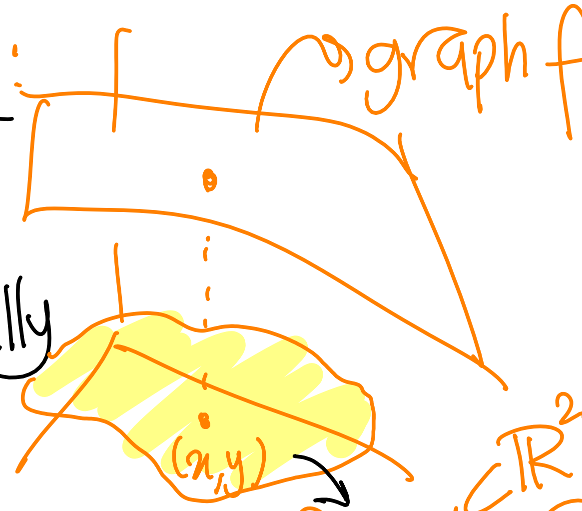
SURFACES & INTEGRALS

§ Parametrization

$S \subset \mathbb{R}^3$ which can be $\underset{(1-1)}{\text{parametrized}} \to C^1$ locally by 2 parameters is called a surface (C^1).

ex:



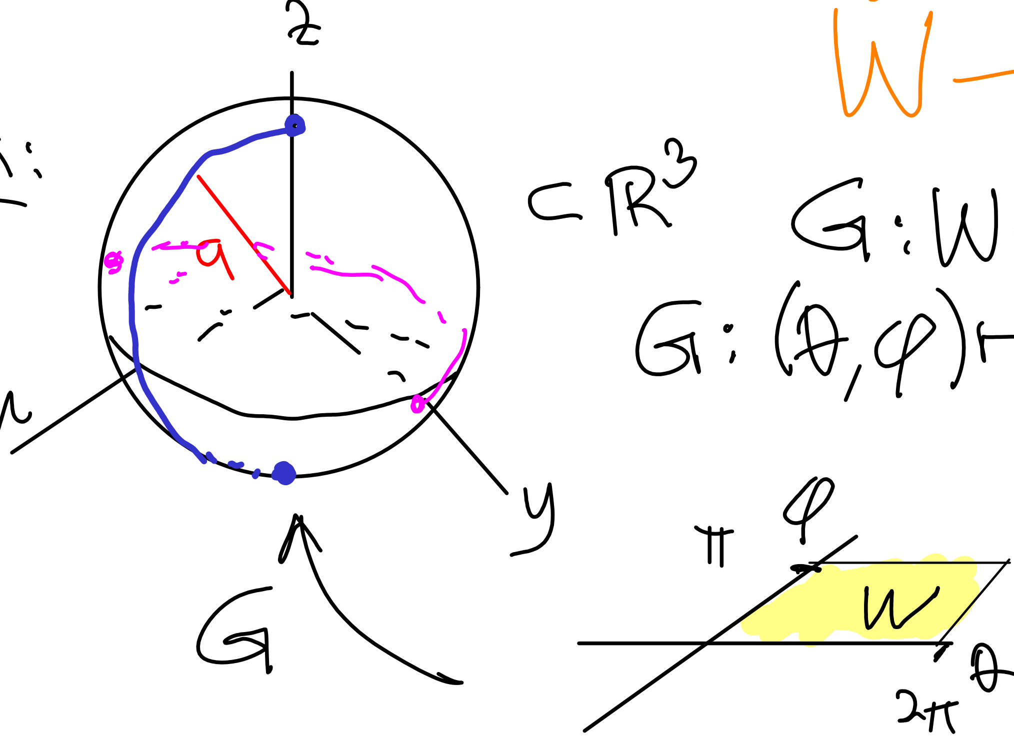
$f: W \to \mathbb{R}$

A param'n for S

$G: (x,y) \mapsto (x, y, f(x,y))$

$W \to \mathbb{R}^3$

ex:



$\subset \mathbb{R}^3$

$G: W \to \mathbb{R}^3$

$G: (\theta, \varphi) \mapsto (x, y, z)$

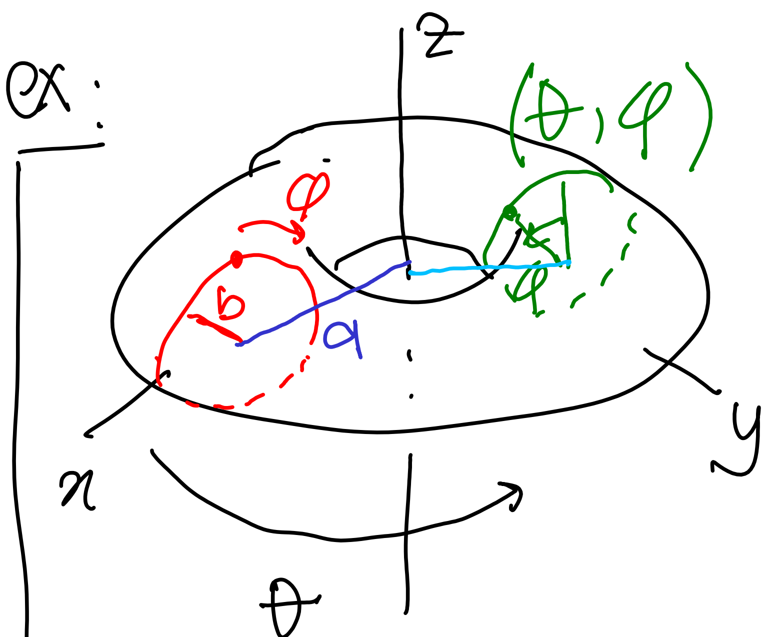
$x = a \sin\varphi \cos\theta$

$y = a \sin\varphi \sin\theta$

$z = a \cos\varphi$

$W = (0, 2\pi) \times (0, \pi)$

ex:



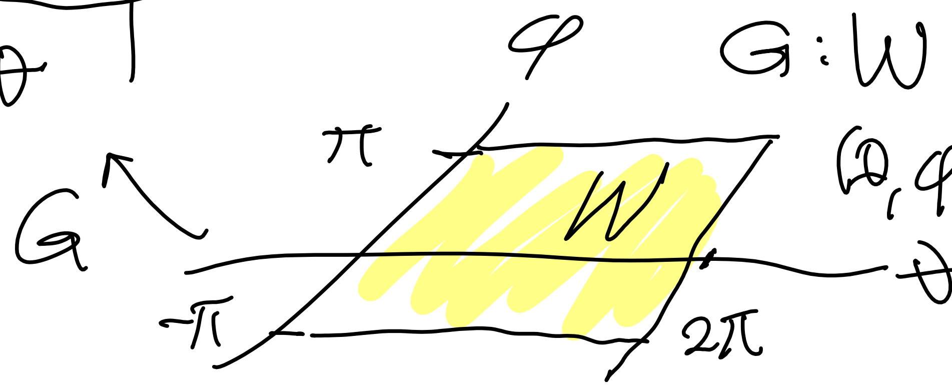
$z = b\cos\varphi$

$x = b\sin\varphi\cos\theta + a\cos\theta$

$y = b\sin\varphi\sin\theta + a\sin\theta$

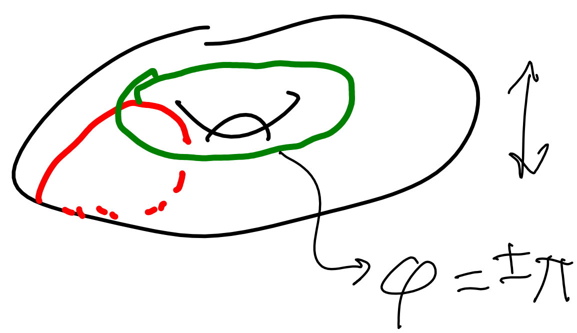
$G : W \to \mathbb{R}^3$

$(\theta, \varphi) \mapsto (x, y, z)$



$W = (0, 2\pi) \times (-\pi, \pi)$

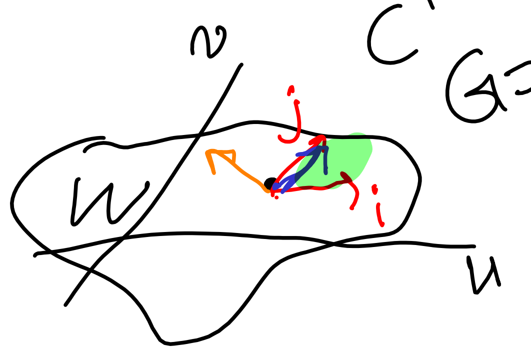
G misses

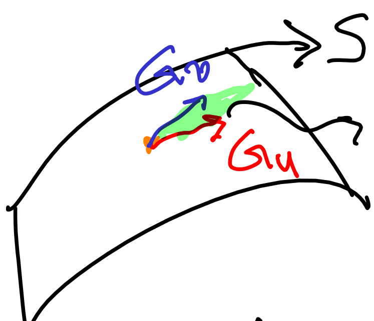


Area of surfaces

C^1

$G = (x, y, z)$





$\text{area} = |G_u \times G_v|$

$$DG(u, v) = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}_{3\times 2}$$

$$DG(u,v)\left(\vec{i}\right) = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix}(u,v) = G_u(u,v)$$

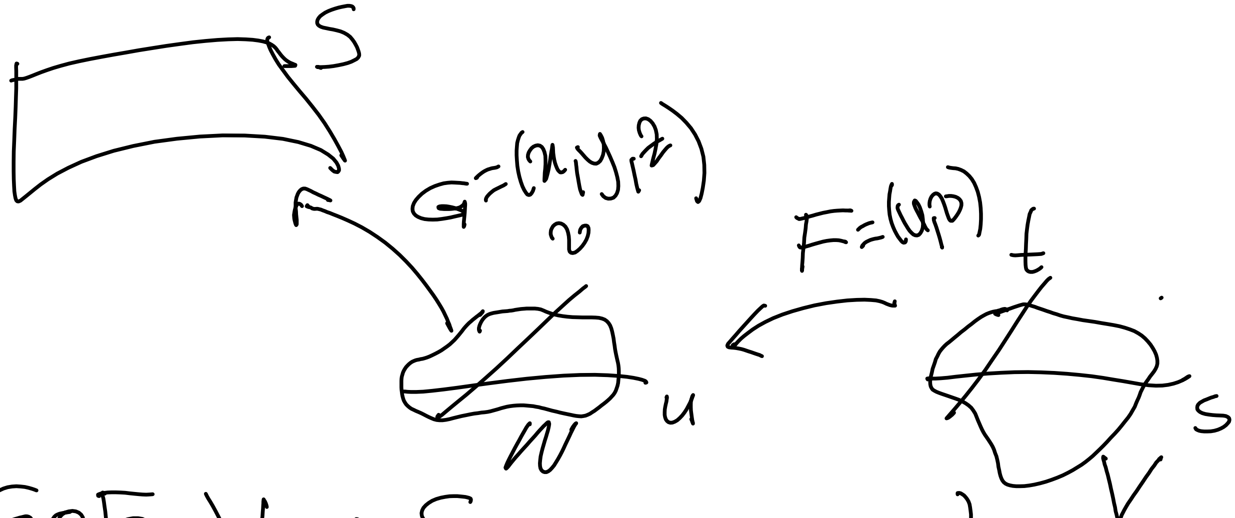
$$\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

<u>defn:</u>

$$\text{area } S = \iint_S dA \overset{\text{defn}}{=} \iint_W \overbrace{\left| G_u(u,v) \times G_v(u,v) \right|}^{\text{cont}} \, du\, dv$$

exists if W is measurable.

- area, if exists, is independent from the parametrization:



$G \circ F : V \to S$: a new param'n.

<u>fact:</u> $$\iint_V \left| (G\circ F)_s \times (G \circ F)_t \right| ds\, dt = \iint_W \left| G_u \times G_v \right| du\, dv$$

<u>proof:</u> $D(G\circ F)(s,t) = DG(\overbrace{F(s,t)}^{(u,v)}) \circ DF(s,t)$ Chain Rule

$$\Leftrightarrow \begin{pmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{pmatrix} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix} \cdot \begin{pmatrix} u_s & u_t \\ v_s & v_t \end{pmatrix} \quad (*)$$

$$G_u \times G_v = (y_u z_v - y_v z_u, \dots, \dots)$$

$$\begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} \times \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix}$$

$$(G \circ F)_s \times (G \circ F)_t = (y_s z_t - y_t z_s, \dots, \dots)$$

The relation btw these two cross products is given by $(*)$.

By Change of Coords the last term is exactly $\underbrace{\text{that}}_{\text{the Jacobian}}$ should appear changing from (s,t) to (u,v).

$\blacksquare$

ex: $S = \text{graph of } h : W \to \mathbb{R}$

$$= \{ \underbrace{(x, y, h(x,y))}_{G : W \to \mathbb{R}^3} : (x,y) \in W \subset \mathbb{R}^2 \}$$

$$G_x = (1, 0, h_x), \quad G_y = (0, 1, h_y)$$

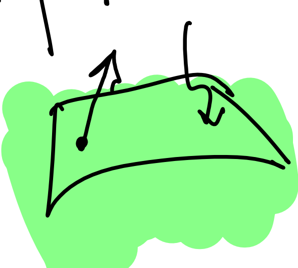
$$G_x \times G_y = (-h_x, -h_y, 1)$$

$$|G_x \times G_y| = \sqrt{1 + h_x^2 + h_y^2}$$

$$\text{area } S = \iint_W \sqrt{1 + h_x^2 + h_y^2}\, dx\,dy$$

§ Integrals over surfaces

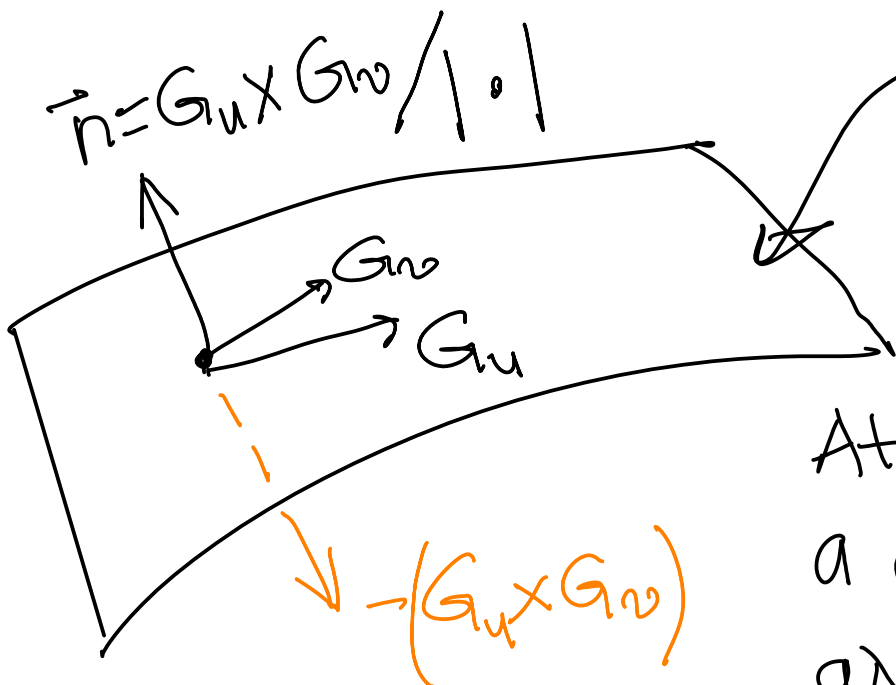
For $f : S \to \mathbb{R}$, $\displaystyle\iint_S f\,dA \overset{\text{defn}}{=} \iint_W f(G(u,v)) \cdot |G_u \times G_v|\, du\,dv$



For given oriented S: $\displaystyle\iint_S \vec{F} \cdot \vec{n}\, dA$ Take G which gives that orientation

with $\vec{n}$ & $\vec{F} : S \to \mathbb{R}$ $\overset{\text{defn}}{=} \displaystyle\iint \vec{F}(G(u,v)) \cdot \vec{n}_{u,v}\, |G_u \times G_v|\, du\,dv$

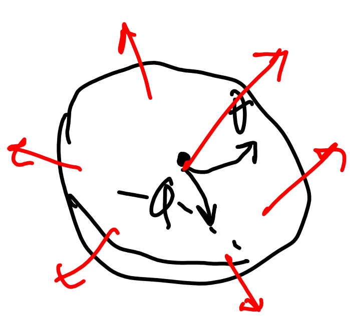
$\vec{n}$ here is a unit normal vector field over S that gives an orientation for S



At any point of S, a chosen param'n gives either a normal vector $\vec{n}$ OR $-\vec{n}$.

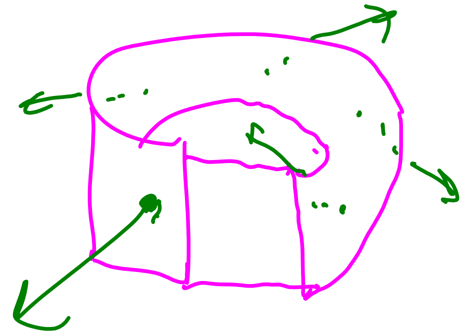
defn: If for every point of S there is such a unit normal vector that are given by param'ns & which vary consistently is called an orientation for the surface.

ex:



(φ, θ) param'n gives an orientation of sphere which is directed outwards fact: A $\left|\begin{matrix}\text{sphere}\\ \text{torus}\end{matrix}\right|$ is orientable;

a Möbius band is not;

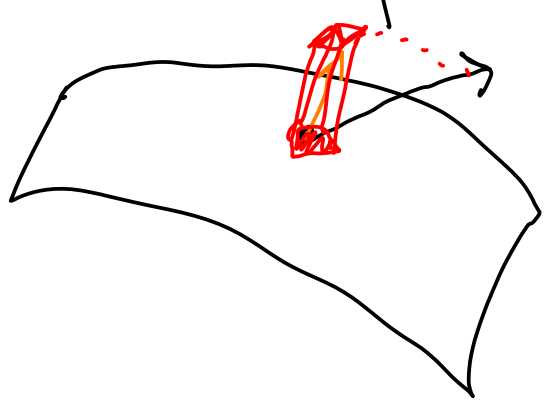


$$\iint_S \vec{F}\cdot\vec{n}\, dA = \iint_W \vec{F}(G(u,v))\cdot\vec{n}_{u,v}\, |G_u \times G_v|\, du\, dv$$

$$\vec{n} = \vec{G}_u \times \vec{G}_v \Big/ |\cdot|$$

$$= \iint_W \vec{F}(G(u,v))\cdot(\vec{G}_u \times \vec{G}_v)\, du\, dv$$

Motivation for $\iint_S \vec{F}\cdot\vec{n}\, dA$:



Let S be a net embedded in a river; $\vec{F}$ be the flow of water. $\iint \vec{F}\cdot\vec{n}$ measures the FLUX of $\vec{F}$ thru S.

thm (Divergence Theorem)

$$\iint_{S=\partial\Omega} \vec{F}\cdot\vec{n}\,dA = \iiint_{\Omega} \operatorname{div}\vec{F}\,dV$$

outward directed unit normal vector field to $\partial\Omega$.

Ω: a nice domain in $\mathbb{R}^3$: S can be parametrized over some measurable $W\subset\mathbb{R}^2$.

e.g. S is piecewise C^1.

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_W \vec{F}(G(u,v))\cdot\vec{n}_{u,v}\,|G_u\times G_v|\,du\,dv$$

$$\vec{n} = \vec{G}_u\times\vec{G}_v \,/\, |\cdot|$$

$$= \iint_W \vec{F}(G(u,v))\cdot(G_u\times G_v)\,du\,dv$$

$$\iint_S f\,dA = \iint_W f(G(u,v))\,|\partial_u G\times\partial_v G|\,du\,dv$$

$W \subset \mathbb{R}^2$

Parametrize with $G(u,v)$, $G: W\to\mathbb{R}^3$

$$\operatorname{grad} f = \vec{\nabla}f = (\partial_x,\partial_y,\partial_z)f = (\partial_x f,\partial_y f,\partial_z f)$$

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = (R_y-Q_z,\ P_z-R_x,\ Q_x-P_y)$$

$\vec{F} = (P,Q,R)$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = (\partial_x,\partial_y,\partial_z)\cdot(P,Q,R) = \partial_x P+\partial_y Q+\partial_z R$$

$$\vec{n} = (n_1, n_2, n_3), \quad |\vec{n}| = 1$$

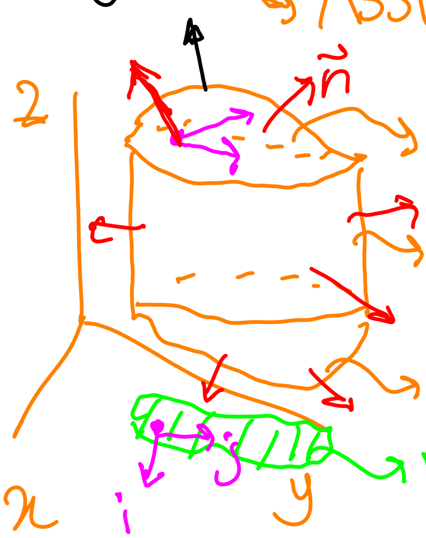
$$\iint_{\partial\Omega} (P, Q, R) \cdot \vec{n}\, dA = \iiint_{\Omega} (\partial_x P + \partial_y Q + \boxed{\partial_z R})\, dV$$

$$= \iint_{\partial\Omega} (P n_1 + Q n_2 + \boxed{R n_3})\, dA$$

Take, for example,

$$\iint_{\partial\Omega} R n_3\, dA = \iint_{S_0 \cup S_1 \cup S_2} = \iint_{S_2} R n_3\, dA + \iint_{S_1} R n_3\, dA$$

$\rightarrow$ Assume:



graph of $\varphi_2 : W \to \mathbb{R}$: S_2

vertical walls: $n_3 = 0$: S_0

graph of $\varphi_1 : W \to \mathbb{R}$: S_1

$\rightarrow W$

W Parametrization

$(x, y) \overset{G}{\mapsto} (x, y, \varphi_2(x, y))$

$(x, y) \mapsto (x, y, \varphi_1(x, y))$

$$S_2 = \{z - \varphi_2(x,y) = 0\} \leftarrow \vec{n}_{S_2} = (-\partial_x \varphi_2, -\partial_y \varphi_2, 1)$$

(downward outward) $\vec{n}_{S_1} = (\partial_x \varphi_1, \partial_x \varphi_1, -1)$

$$= \iint_W R(x,y,\varphi_2(x,y))\cdot 1\cdot dx\,dy$$
$$+ \iint_W R(x,y,\varphi_1(x,y))\cdot(-1)\cdot dx\,dy$$

$$= \iint_W \Big(R(x,y,\varphi_2) - R(x,y,\varphi_1)\Big)\,dx\,dy$$

$$\underset{\text{FTC}}{=} \iint_W \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} \partial_z R(x,y,z)\,dz\;dx\,dy$$

$$\underset{\text{Fubini}}{=} \iiint_\Omega \partial_z R\,dV$$

Similarly for the other two terms with x-simplicity or y-simplicity. $\blacksquare$

Remark: $$\iint_S \vec{F}\cdot\vec{n}\,dA = \iiint_\Omega \operatorname{div}\vec{F}\,dV \approx \operatorname{div}\vec{F}(a)\,\tfrac{4}{3}\pi\varepsilon^3$$

flux over small sphere / vol(ball) $= \operatorname{div} F(a) \sim$ the source at a.