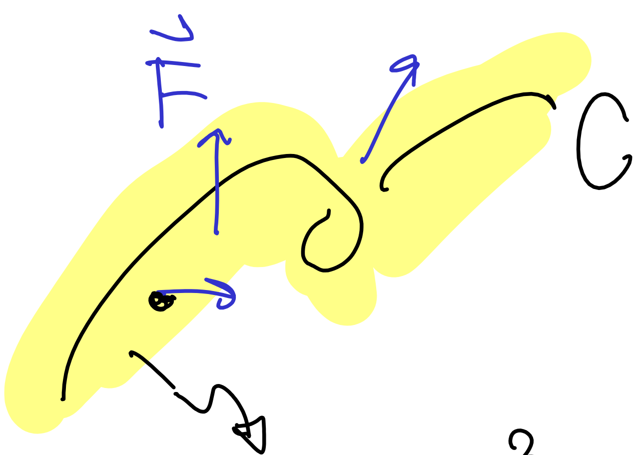
§ Work integral

One can define line integrals of vector fields along curves:



$F: U \to \mathbb{R}^3$

Everything in red is just notation! They make sense when you fix a param'n.

$$\int_a^b \vec{F} \cdot d\vec{x} = \int_a^b \vec{F} \cdot (dx_1, \dots, dx_n)$$

$$\vec{F} = (f_1, \dots, f_n)$$

$$= \int_a^b f_1 dx_1 + \dots + f_n dx_n$$

$$\overset{\text{defn}}{=} \int_a^b f_1(t) \cdot x_1'(t)\, dt + \dots + \int_a^b f_n(t)\, x_n'(t)\, dt$$

$$= \int_a^b (f_1, \dots, f_n) \cdot (x_1', \dots, x_n')\, dt$$

$$= \int_a^b \vec{F}(g(t)) \cdot \vec{g}\,'(t)\, dt = \int_C \vec{F} \cdot d\vec{x}$$

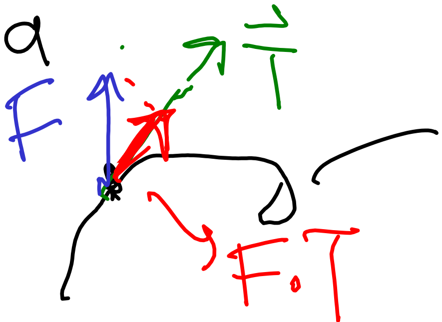
Take param'n $\vec{g}$.

$$= \int_a^b \vec{F}(g(t)) \cdot \frac{\vec{g}\,'(t)}{|\vec{g}\,'(t)|} \cdot |\vec{g}\,'(t)|\, dt$$

defn: $\vec{g}\,'(t)$ is called a tangent vector to C at point $\vec{g}(t)$.

$T(t) = \vec{g}\,' / |g'|$ is called a unit tangent vector.

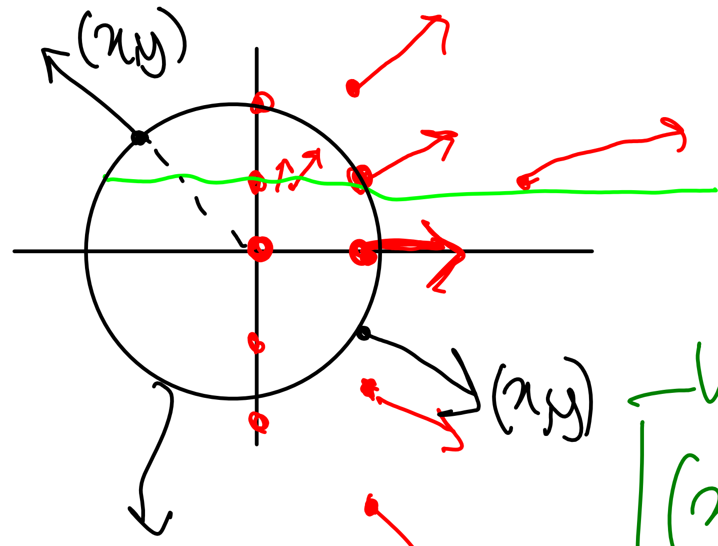
$$= \int_a^b \vec{F} \cdot \vec{T}(t) \cdot |g'(t)|\, dt$$



$$= \int_a^b \vec{F}_{tangential} \cdot |g'(t)|\, dt$$

The line integral $\vec{F}$ over C is called the **work integral** too.

ex: $\vec{F}(x,y) = (x^2, xy)$.



C: unit circle

$F(1,y) = (1,y)$

$F(x,1) = (x^2, x)$

Work is zero because

$(x^2, xy)\cdot(-y, x) = 0$:

$\vec{F} \perp$ tangent direction

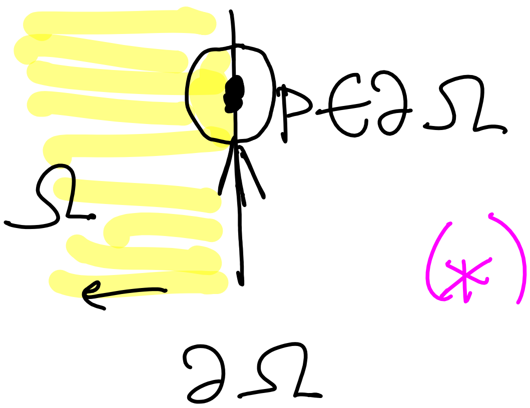
$$\int_C \vec{F}\cdot d\vec{x} = \int_0^{2\pi} (x^2, xy)\cdot(-St, Ct)\,dt$$

$g(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$

$$= \int_0^{2\pi} (-C^2tSt + C^2tSt)\,dt = 0$$

Green's Theorem (in $\mathbb{R}^2$)

Orientation of $\partial\Omega$ wrt Ω:



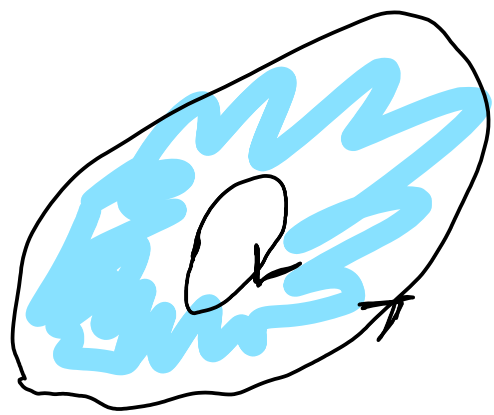
(*) For this defn we assume $\text{int}\,\Omega \neq \emptyset$ & every $p \in \partial\Omega$ has an int point in each of its nbhd.

<u>defn</u>: A cpct domain $\Omega \subset \mathbb{R}^2$ satisfying (*) is called <u>regular</u>.

<u>ex</u>:

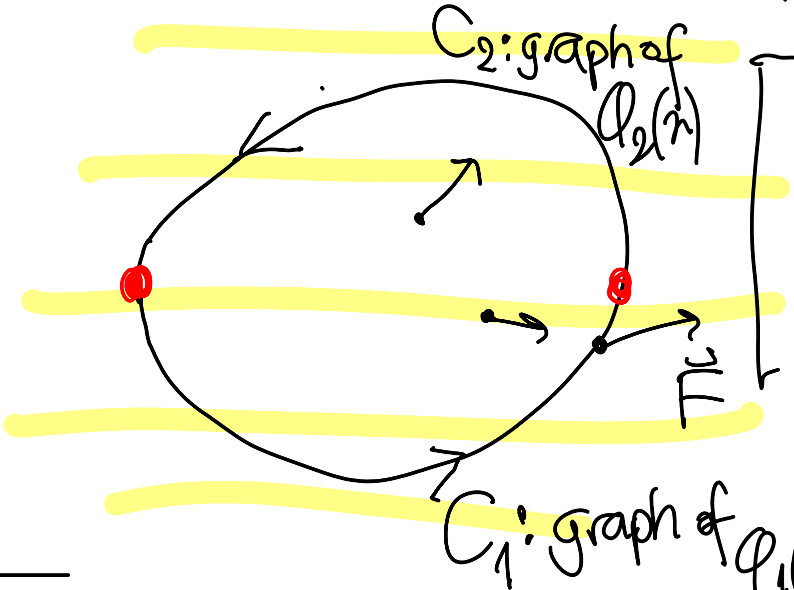
regular

counterclockwise orientation is THE one coming from the domain



<u>defn</u>: Closed curve is the one which starts & ends at the same point. Simple curve does not intersect itself.

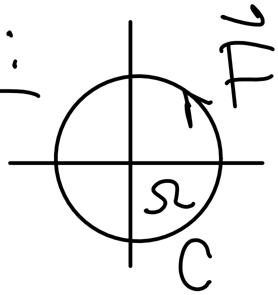
thm: Let Ω be regular; $\partial\Omega$ a simple closed curve (not necessarily connected) that is piecewise C^1; $\vec{F} = (P(x,y), Q(x,y))$ a C^1 vector field in a nbhd of Ω. Then



$$\boxed{\int_{\partial\Omega} \vec{F}\cdot d\vec{x} = \iint_{\Omega} (Q_x - P_y)\, dA}$$

$\partial\Omega$ oriented wrt Ω

ex:



$$\vec{F}(x,y) = \Big(\underbrace{\sqrt{1+x^2} - y e^{xy} + 3y}_{P},\ \underbrace{x^2 - x e^{xy} + \log(1+y^4)}_{Q}\Big)$$

$$\int_C \vec{F}\cdot d\vec{x} \underset{\text{Green's}}{=} \iint_{\Omega} \Big[\big(2x - \cancel{e^{xy}} - \cancel{xye^{xy}}\big) - \big(-\cancel{e^{xy}} - \cancel{xye^{xy}} + 3\big)\Big]\, dA$$

$$= \iint_{\Omega} (2x - 3)\, dA = -3\pi$$

brings 0 thx to symmetry

<u>proof of thm:</u> $C = \text{graph of } \varphi_1 \cup \text{graph of } \varphi_1.$

$C_1 : x \mapsto (x, \varphi_1(x))$ (with right orientation)

$C_2 : x \mapsto (x, \varphi_2(x))$ (w/ wrong ")

$(x \in [a,b])$

$$\iint_\Omega \frac{\partial P}{\partial y}(x,y)\,dA \underset{\text{Fubini}}{=} \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}(x,y)\,dy \right) dx$$

$$\overset{\text{FTC}}{=} \int_a^b \Big(P(x, \varphi_2(x)) - P(x, \varphi_1(x)) \Big) dx$$

$$= - \left[\int_a^b P(x, \varphi_1)\,dx - \int_a^b P(x, \varphi_2)\,dx \right]$$

$$= - \left(\int_{C_1} P(x,y)\,dx + \int_{C_2} P(x,y)\,dx \right) = - \int_C P\,dx$$

Parametrize C_1: $(t, \varphi_1(t))$ $t \in [a,b]$

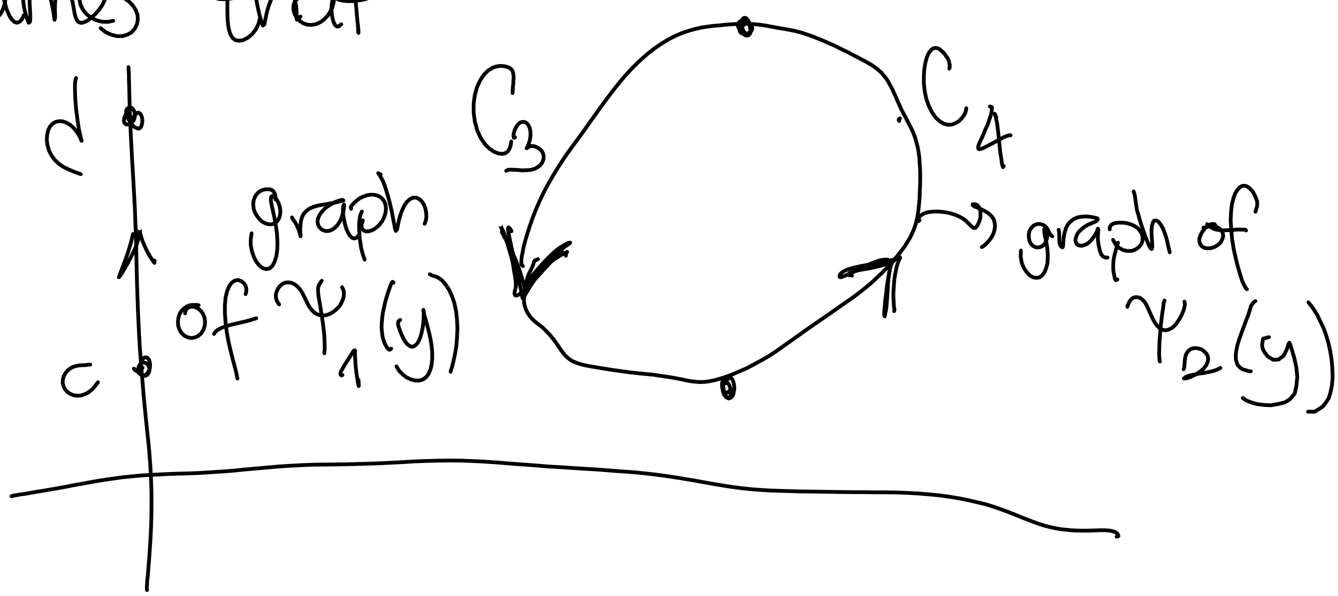
$$= \int_a^b P(x, \varphi_1(x)) \underbrace{\frac{dx}{dt}}_{= dt/dt = 1} \cdot dt$$

Recall:

$$\int_{\partial\Omega} \tilde{F} \cdot dx = \int_{\partial\Omega} P\,dx + Q\,dy$$

Similarly: $\iint_{\Omega} Q_x \, dA = \int_{\partial\Omega} Q\,dy$ if one

assumes that



$-$ sign does not appear now because $(\psi_1(y), y)$ from c to d gives the wrong OR for C_3 but $(\psi_2(y), y)$ gives the right one for C_4.

Observe:

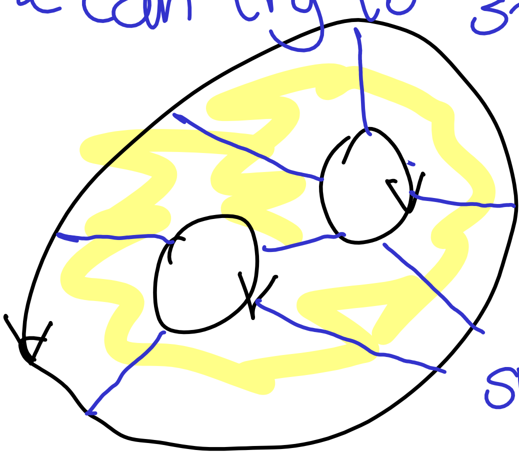
① If $\vec{F} = \vec{\nabla} f$ for some $f : \Omega \to \mathbb{R}$, C^2

$$\int_{\partial\Omega} \vec{F} \cdot d\vec{x} = \iint_{\Omega} \left[\frac{\partial}{\partial x}(f_y) - \frac{\partial}{\partial y}(f_x) \right] dA = \iint_{\Omega} (f_{yx} - f_{xy})\, dA$$

$$= 0.$$

② Our proof is very restrictive.

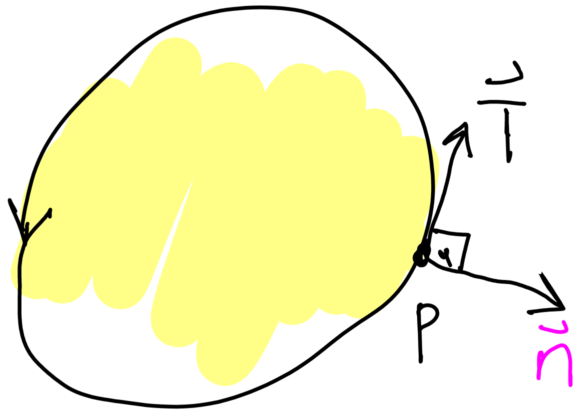
For general case, one can try to split the domain into nice subdomains that fit in our proof.



But in general that does not work (Appendix)

③ $\displaystyle\int_{\partial\Omega} \vec{F}\cdot\vec{n}\,ds = \int_{\partial\Omega} (P,Q)\cdot\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\vec{T}\,ds$

$$= \int_{\partial\Omega} (-Q,P)\cdot T\,ds$$



$$\overset{\text{Green's}}{=} \iint_{\Omega} (P_x + Q_y)\,dA$$

$\vec{n}$: outward directed

$\vec{P} + \varepsilon\vec{n} \in \Omega$

$$\vec{n} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\vec{T}$$

$$= \iint_{\Omega} \vec{\nabla}\cdot\vec{F}\,dA$$

where $\vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\cdot(P,Q) = P_x + Q_y$

the divergence of F.