Group extensions in second order jet group

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Abstract

The second order jet group $\mathcal{G}_2$ projects onto $\mathcal{G}_1$ with kernel $\mathcal{K}_2$. We denote this exact sequence by Φ and its Lie algebra analogue by $\mathcal{L}\Phi$. For a Lie group $\mathcal{G} \subset \mathcal{G}_1$ and a kernel $\mathcal{V} \subset \mathcal{K}_2$ for $\mathcal{G}$, we define the vector space $H^2(\mathcal{G}, \mathcal{V}, \Phi)$ of extensions of $\mathcal{G}$ by $\mathcal{V}$ restricted to Φ and its Lie algebra analogue $H^2(\mathfrak{g}, \mathcal{V}, \mathcal{L}\Phi)$. We give examples where dim $H^2(\mathfrak{g}, \mathcal{V}, \mathcal{L}\Phi) \neq 0$ whereas dim $H^2(\mathfrak{g}, \mathcal{V}) = 0$.

1. Introduction

Let $\mathcal{G}_i(n) = GL_i(n, \mathbb{R})$ be the set of i-jets $j^i(f)_0$ of local diffeomorphisms f with source and target at the origin $0 \in \mathbb{R}^n$ where $i = 1, 2$ in this paper. $\mathcal{G}_i(n)$ is a Lie group with the group operation induced by composition of jets: $j^i(f)_0 \circ j^i(g)_0 = j^i(f \circ g)_0$. We have the projection homomorphism $\pi : \mathcal{G}_2(n) \to \mathcal{G}_1(n)$ with kernel $\mathcal{K}_2(n)$ which is a vector space. Let $\mathcal{S} \subset \mathcal{G}_2(n)$ be a Lie subgroup such that the kernel $\mathcal{V}$ of $\pi|_{\mathcal{S}}$ is a subspace of $\mathcal{K}_2(n)$. Let $\mathcal{G} = \pi(\mathcal{S}) \subset \mathcal{G}_1(n)$.

In this paper, we will be concerned with the following question

QA : Can $\mathcal{S}$ contain any information not contained in $\mathcal{G}$?

For instance, the homotopy exact sequence of the fibration $\mathcal{V} \to \mathcal{S} \to \mathcal{G}$ shows that $\pi_i(\mathcal{S}) = \pi_i(\mathcal{G}),\ i = 1, 2, \ldots,$ so that $\mathcal{S}$ does not contain any new information from topological viewpoint. On the other hand, it follows from [Ter78] that $\mathcal{S}$ may have representations which are not obtained by prolonging a representation of $\mathcal{G}$.

We may now ask the global analogue of **QA**: Let M be a differentiable manifold with dim $M = n$ and $P(M, \mathcal{S}) \to M$ be an $\mathcal{S}$-structure on M, i.e. $P(M, \mathcal{S})$ is a reduction of the structure group $\mathcal{G}_2(n)$ of the second order frame bundle $F(M, \mathcal{G}_2(n)) \to M$ to $\mathcal{S}$. We have the projection map $\pi : P(M, \mathcal{S}) \to P(M, \mathcal{G}) \subset F(M, \mathcal{G}_1(n))$ = first order frame bundle of M.

QB : Can $P(M, \mathcal{S})$ contain any information not contained in $P(M, \mathcal{G})$?

We can incorporate the data given in the first paragraph into the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{S} & \xrightarrow{\pi|_{\mathcal{S}}} & \mathcal{G} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K}_2(n) & \longrightarrow & \mathcal{G}_2(n) & \xrightarrow{\pi} & \mathcal{G}_1(n) & \longrightarrow & 1 \end{array} \qquad (1)$$

where the vertical maps are inclusions.

This paper is organized as follows. In Section 2, we assume that all groups in (1) are abstract groups and denote the lower sequence in (1) by Φ (dropping n also in the spaces in the lower sequence in (1)). The following assumptions on Φ will be made during the course of this paper.

A1 : Φ splits, i.e. $[\mathcal{G}_2] = 0$ in $H^2(\mathcal{G}_1, \mathcal{K}_2)$.

A2 : $\mathcal{K}_2$ acts transitively on the splittings of Φ, i.e. $H^1(\mathcal{G}_1, \mathcal{K}_2) = 0$.

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With these assumptions we give the construction of the vector space $H^2(\mathcal{G},\mathcal{V},\Phi)$ whose elements are equivalence classes of extensions contained in Φ as in (1), modulo the action of $\mathcal{K}_2$ (see Propositions 4, 5 and Definition 6). The main point is that the definition of $H^2(\mathcal{G},\mathcal{V},\Phi)$ does not use any concept outside Φ.

In Section 3, we recall the definition of the second order jet group $\mathcal{G}_2(n)$ and the lower sequence in (1) which we denote by $\Phi(n)$. We show that $\Phi(n)$ satisfies **A1** and **A2** and therefore $H^2(\mathcal{G},\mathcal{V},\Phi(n))$ is defined.

In Section 4, we define the Lie algebra analogue $H^2(\mathcal{LG},\mathcal{V},\mathcal{L}\Phi(n))$ and define the homomorphism $H^2(\mathcal{G},\mathcal{V},\Phi(n))\to H^2(\mathcal{LG},\mathcal{V},\mathcal{L}\Phi(n))$. Thus we have the commutative diagram

$$\begin{array}{ccc} H^2(\mathcal{G},\mathcal{V},\Phi(n)) & \longrightarrow & H^2(\mathcal{LG},\mathcal{V},\mathcal{L}\Phi(n)) \\ \downarrow & & \downarrow \\ H^2(\mathcal{G},\mathcal{V}) & \longrightarrow & H^2(\mathcal{LG},\mathcal{V}) \end{array} \tag{2}$$

The vertical homomorphisms in (2) are induced by inclusion. The cohomology groups $H^*(\mathfrak{g},\mathcal{V})$ are defined and studied first in [CE48] for an arbitrary Lie algebra $\mathfrak{g}$ and representation $\mathcal{V}$. The group $H^2(\mathcal{G},\mathcal{V})$ is defined (again for an arbitrary Lie group $\mathcal{G}$ and representation $\mathcal{V}$) and studied in detail in [Hoch51] (see [HM62] for $H^*(\mathcal{G},\mathcal{V})$). It is shown in [Hoch51] that the lower homomorphism in (2) is an isomorphism if $\mathcal{G}$ is simply connected. This isomorphism is generalized to $H^*(\mathcal{G},\mathcal{V})\to H^*(\mathcal{LG},\mathcal{V})$ in [VEst53] with the assumption that $\mathcal{G}$ is homologically trivial in certain dimensions.

In Section 5, we give an affirmative answer to **QA** by constructing explicit examples where $\dim H^2(\mathcal{LG},\mathcal{V},\mathcal{L}\Phi(n))\neq 0$ whereas $\dim H^2(\mathcal{LG},\mathcal{V})=0$ (e.g. see Proposition 15). This fact shows that we may loose information in general when we pass from the upper row to the lower row in (2) which incorporates continuous cohomology. Our examples raise a plethora of questions in relation to higher order geometries which we hope to study in some future work (see [CSS01] and the references therein for an important class of such geometries, called parabolic geometries).

In Section 6, we make some remarks on the possible global consequences of the present framework based on [GOO1] whose motivation is to give a positive answer also to **QB**. Finally we would like to indicate that the present work, [GOO1] and [GOO2] should be considered as the initial steps of a program set forth in [Ort06] which proposes a possible generalization of Klein's Erlangen program.

2. Construction of the vector space $H^2(\mathcal{G},\mathcal{V},\Phi)$

Let $\mathcal{G}_1$ be an abstract group with identity $\mathbb{1}$ and $\mathcal{K}$ be a vector space. Let $\mathcal{G}_2$ be an extension of $\mathcal{G}$ by $\mathcal{K}_1$, i.e. we have the exact sequence

$$\Phi: \quad 0\longrightarrow\mathcal{K}\longrightarrow\mathcal{G}_2\xrightarrow{\pi}\mathcal{G}_1\longrightarrow 1 \tag{3}$$

Φ will be the same throughout this section.

Recall that (3) gives an action of $\mathcal{G}_1$ on $\mathcal{K}$ defined as follows: For $g_1\in\mathcal{G}_1$, we choose some $g_2\in\mathcal{G}_2$ with $\pi(g_2)=g_1$ and define $g_1k\doteq g_2k(g_2)^{-1}$. Since $\mathcal{K}$ is abelian, g_1k does not depend on g_2 and we have $g_1(k+k')=g_1k+g_1k'$.

We now make the following assumption

A1 : There exists a homomorphism $\sigma:\mathcal{G}_1\to\mathcal{G}_2$ such that $\pi\circ\sigma=id$, in other words, Φ splits.

For any $g_2\in\mathcal{G}_2$, we have $g_2=k\sigma(g_1)$ with $k=g_2(\sigma\circ\pi(g_2^{-1}))\in\mathcal{K}$ and $g_1=\pi(g_2)$. Also, if $g_2=k\sigma(g_1)=k'\sigma(g_1')$ then $\sigma(g_1'g_1^{-1})=k'k^{-1}$ so that $g_1'=g_1$, $k=k'$, thus the representation $g_2=k\sigma(g_1)$ is unique. Let $\mathcal{G}_1\propto\mathcal{K}$ be the semidirect product, i.e. $\mathcal{G}_1\propto\mathcal{K}$ is $\mathcal{G}_1\times\mathcal{K}$ as a set and the group operation is $(g_1,k)(g_1',k')=(g_1g_1',k+g_1k')$. We have the isomorphism $\mathcal{G}_2\to\mathcal{G}_1\propto\mathcal{K}$ defined by $k\sigma(g_1)\to(g_1,k)$, i.e. σ gives an identification of $\mathcal{G}_2$ with $\mathcal{G}_1\propto\mathcal{K}$.

We now fix σ once and for all until further notice. We will use both notations (g_1, k) and $k\sigma(g_1)$ (strictly speaking, we should write $(g_1, k)_\sigma$ for (g_1, k)).

Definition 1. *Let $\mathcal{G}$be a subgroup of $\mathcal{G}_1$. A subspace $\mathcal{V}$ of $\mathcal{K}$ is called a kernel for $\mathcal{G}$ if there exists a subgroup $\mathcal{S}$of $\mathcal{G}_2$ with $\pi(\mathcal{S}) = \mathcal{G}$ and $\mathcal{V} = \ker(\pi|_{\mathcal{S}})$.*

Thus we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{S} & \xrightarrow{\pi|_{\mathcal{S}}} & \mathcal{G} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{G}_2 & \xrightarrow{\pi} & \mathcal{G}_1 & \longrightarrow & 1 \end{array} \tag{4}$$

where the vertical maps are inclusions. Note that $\mathcal{G}$ acts on $\mathcal{V}$ and this action is compatible with the action of $\mathcal{G}_1$ on $\mathcal{K}$.

Definition 2. *A commutative diagram of the form (4) is called an extension of $\mathcal{G}$ by $\mathcal{V}$ restricted to Φ. We will denote (4) simply by $\mathcal{S}$ and call $\mathcal{S}$ a (restricted) extension (of $\mathcal{G}$ by $\mathcal{V}$) and denote the set of all restricted extensions by $\mathcal{H}(\mathcal{G}, \mathcal{V}, \Phi)$.*

Our purpose is to endow $\mathcal{H}(\mathcal{G}, \mathcal{V}, \Phi)$ with a vector space structure over $\mathbb{R}$. First we observe that the Baer sum of two restricted extensions $\mathcal{S}$ and $\mathcal{S}'$ can be viewed as another restricted extension: Note that $\Delta_{\mathcal{S},\mathcal{S}'} \doteq \{(s, s') \in \mathcal{S} \times \mathcal{S}' \mid \pi(s) = \pi(s')\}$ is a subgroup of the product $\mathcal{S} \times \mathcal{S}'$. Using the fact that both extensions $\mathcal{S}$ and $\mathcal{S}'$ realize the same action of $\mathcal{G}$ on $\mathcal{V}$, we see that $(\mathcal{V}, -\mathcal{V})$ is a normal subgroup of $\Delta_{\mathcal{S},\mathcal{S}'}$, consisting of elements of the form $(v, -v),\ v \in \mathcal{V}$. As well known, we construct the Baer sums $\mathcal{S} \oplus \mathcal{S}' \doteq \Delta_{\mathcal{S},\mathcal{S}'}/(\mathcal{V}, -\mathcal{V})$ and $\mathcal{G}_2 \oplus \mathcal{G}_2 \doteq \Delta_{\mathcal{G}_2,\mathcal{G}_2}/(\mathcal{K}, -\mathcal{K})$. Note that the inclusion $i : \mathcal{S} \times \mathcal{S}' \to \mathcal{G}_2 \times \mathcal{G}_2$ injects $\mathcal{S} \oplus \mathcal{S}'$ into $\mathcal{G}_2 \oplus \mathcal{G}_2$ as a subgroup. Upon identifying $\mathcal{G}_2$ by $\mathcal{G}_1 \propto \mathcal{K}$ via σ, we now define $\overline{\sigma} : \mathcal{G}_2 \oplus \mathcal{G}_2 \to \mathcal{G}_2$ by

$$\overline{\sigma}\{(g_2, k), (g_2, k')\} = (g_2, k + k')$$

where $\{(g_2, k), (g_2, k')\}$ denotes the coset of $((g_2, k), (g_2, k'))$ in $\Delta_{\mathcal{G}_2,\mathcal{G}_2}$. It can be verified without difficulty that $\overline{\sigma}$ is well-defined and an isomorphism. Thus, we obtain the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{S} \oplus \mathcal{S}' & \longrightarrow & \mathcal{G} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow i & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{G}_2 \oplus \mathcal{G}_2 & \longrightarrow & \mathcal{G}_2 & \longrightarrow & 1 \\ & & \| & & \downarrow \overline{\sigma} & & \| & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \mathcal{G}_2 & \longrightarrow & 1 \end{array}$$

Now we define:

$$\mathcal{S} \overset{\sigma}{\oplus} \mathcal{S}' \doteq (\overline{\sigma} \circ i)(\mathcal{S} \oplus \mathcal{S}') \in \mathcal{H}(\mathcal{G}, \mathcal{V}, \Phi). \tag{5}$$

Alternatively, we can define $\mathcal{S} \overset{\sigma}{\oplus} \mathcal{S}'$ by giving its elements in the fiber over $g \in \mathcal{G}$:

$$\pi\,|^{-1}_{\mathcal{S}\overset{\sigma}{\oplus}\mathcal{S}'}\,(g) \doteq \{kk'\sigma(g) \mid k\sigma(g) \in \pi\,|^{-1}_{\mathcal{S}}\,(g),\ k'\sigma(g) \in \pi\,|^{-1}_{\mathcal{S}'}\,(g)\} \tag{6}$$

where we used multiplicative notation for the operation of $\mathcal{K}$. We clearly have $\mathcal{S} \overset{\sigma}{\oplus} \mathcal{S}' = \mathcal{S}' \overset{\sigma}{\oplus} \mathcal{S}$ and $(\mathcal{S} \overset{\sigma}{\oplus} \mathcal{S}') \overset{\sigma}{\oplus} \mathcal{S}'' = \mathcal{S} \overset{\sigma}{\oplus} (\mathcal{S}' \overset{\sigma}{\oplus} \mathcal{S}'')$.

We now take the particular extension $\overset{\sigma}{\mathcal{E}} \doteq \{(g, v) \mid g \in \mathcal{G}, v \in \mathcal{V}\}$ associated to the splitting σ of Φ whence we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V} & \longrightarrow & \overset{\sigma}{\mathcal{E}} & \underset{\pi}{\overset{\sigma|\mathcal{G}}{\leftrightarrows}} & \mathcal{G} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{G}_2 & \underset{\pi}{\overset{\sigma}{\leftrightarrows}} & \mathcal{G}_1 & \longrightarrow & 1 \end{array} \tag{7}$$

and $\sigma|_{\mathcal{G}}$ splits the upper row of (7), that is, both rows of (7) split in a compatible way. Note that if $s \in \mathcal{S}$, where $\mathcal{S}$ is an arbitrary extension, then $s = (g,k)$, where $g \in \mathcal{G}$ and $k \in \mathcal{K}$ but we might not have $k \in \mathcal{V}$.

Let $(g,k) \in \mathcal{S}$ for some extension $\mathcal{S}$, and $v \in \mathcal{V}$. Since $(\mathbb{1}, v') \in \mathcal{S}$ for all $v' \in \mathcal{V}$, in particular we have $(\mathbb{1}, g^{-1}v) \in \mathcal{S}$ which gives $(g,k)(\mathbb{1}, g^{-1}v) = (g, k+v) \in \mathcal{S}$. This fact together with the definitions of $\overline{\sigma}$ and $\overset{\sigma}{\oplus}$ gives

$$\mathcal{S} \overset{\sigma}{\oplus} \overset{\sigma}{\mathcal{E}} = \mathcal{S}$$

for all $\mathcal{S} \in \mathcal{H}(\mathcal{G}, \mathcal{V}, \Phi)$.

Now for $\mathcal{S} \in \mathcal{H}(\mathcal{G}, \mathcal{V}, \Phi)$ we define $\overset{\sigma}{-}\mathcal{S} \doteq \{(g,-k) \mid (g,k) \in \mathcal{S}\}$. If $(g,k), (g,k') \in \mathcal{S}$, then $(g,k)(g,k')^{-1} = (g,k)(g^{-1}, -g^{-1}k') = (\mathbb{1}, k-k') \in \mathcal{S}$ and hence $k - k' \in \mathcal{V}$. Therefore

$$\mathcal{S} \overset{\sigma}{\oplus} (\overset{\sigma}{-}\mathcal{S}) = \overset{\sigma}{\mathcal{E}}.$$

We now define

$$t \overset{\sigma}{\cdot} \mathcal{S} \doteq \{(g,tk) \mid (g,k) \in \mathcal{S}\} \in \mathcal{H}(\mathcal{G}, \mathcal{V}, \Phi). \tag{8}$$

for $t \in \mathbb{R}$. Clearly, we have $t \overset{\sigma}{\cdot} (\mathcal{S} \overset{\sigma}{\oplus} \mathcal{S}') = (t \overset{\sigma}{\cdot} \mathcal{S}) \overset{\sigma}{\oplus} (t \overset{\sigma}{\cdot} \mathcal{S}')$. The remaining axioms being easily verified, we see that $\mathcal{H}(\mathcal{G}, \mathcal{V}, \Phi)$ is a vector space over $\mathbb{R}$ with the operations defined by (5) and (8). We denote this vector space by $\mathcal{H}_\sigma(\mathcal{G}, \mathcal{V}, \Phi)$.

We will now define an equivalence relation on $\mathcal{H}_\sigma(\mathcal{G}, \mathcal{V}, \Phi)$. Let I_k denote the inner automorphism of $\mathcal{G}_2$, for $k \in \mathcal{K}$, i.e. $I_k(g_2) = k(g_2)k^{-1}$. Note that I_k acts as identity on $\mathcal{V}$ and commutes with the projection π.

Definition 3. *Let $\mathcal{S}, \mathcal{S}' \in \mathcal{H}_\sigma(\mathcal{G}, \mathcal{V}, \Phi)$. Then $\mathcal{S} \sim \mathcal{S}'$ if $I_k(\mathcal{S}) = \mathcal{S}'$ for some $k \in \mathcal{K}$. In this case we have the following commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{S} & \overset{\pi}{\longrightarrow} & \mathcal{G} & \longrightarrow & 1 \\ & & \| & & \downarrow I_k & & \| & & \\ 0 & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{S}' & \overset{\pi}{\longrightarrow} & \mathcal{G} & \longrightarrow & 1 \end{array}$$

We define $H^2_\sigma(\mathcal{G}, \mathcal{V}, \Phi) \doteq \mathcal{H}_\sigma(\mathcal{G}, \mathcal{V}, \Phi)/\sim$ and denote the equivalence class of $\mathcal{S}$ by $[\mathcal{S}]$.

Proposition 4. *The operations*

$$\begin{aligned} [\mathcal{S}] \overset{\sigma}{\oplus} [\mathcal{S}'] &\doteq [\mathcal{S} \overset{\sigma}{\oplus} \mathcal{S}'] \\ t \overset{\sigma}{\cdot} [\mathcal{S}] &\doteq [t \overset{\sigma}{\cdot} \mathcal{S}] \end{aligned}$$

are well-defined and $H^2_\sigma(\mathcal{G}, \mathcal{V}, \Phi)$ is a vector space with $0 = [\overset{\sigma}{\mathcal{E}}]$.

Proof. Suppose that $I_k(\mathcal{S}) = \mathcal{L}$ and $I_{k'}(\mathcal{S}') = \mathcal{L}'$. We define $\lambda_{k,k'} : \Delta_{\mathcal{S},\mathcal{S}'} \to \Delta_{\mathcal{L},\mathcal{L}'}$ by $\lambda_{k,k'}(x,y) = (I_k(x), I_{k'}(y))$. Since $(\mathbb{1},k)(\mathbb{1},v)(\mathbb{1},k)^{-1} = (\mathbb{1}, v)$ and similarly for k', the map $\lambda_{k,k'}$ is identity on $(\mathcal{V}, -\mathcal{V})$ and therefore induces a map $\lambda_{k,k'} : \mathcal{S} \oplus \mathcal{S}' \to \mathcal{L} \oplus \mathcal{L}'$ defined by $\lambda_{k,k'}\{(g,m),(g,m')\} = \{I_k(g,m), I_{k'}(g,m')\}$. Now

$$\begin{aligned} (i \circ \overline{\sigma} \circ \lambda_{k,k'})\{(g,m),(g,m')\} &= i \circ \overline{\sigma}\{I_k(g,m), I_{k'}(g,m')\} \\ &= i \circ \overline{\sigma}\{(g, k+m-gk), (g, k'+m'-gk')\} \\ &= (g, k+k'+m+m'-g(k+k')) \\ &= (\mathbb{1}, k+k')(g, m+m')(\mathbb{1}, k+k')^{-1} \\ &= (I_{k+k'} \circ i \circ \overline{\sigma})\{(g,m),(g,m')\} \end{aligned}$$

which proves $[\mathcal{S} \overset{\sigma}{\oplus} \mathcal{S}'] = [\mathcal{L} \overset{\sigma}{\oplus} \mathcal{L}']$. Also, if $I_k(\mathcal{S}) = \mathcal{L}$ then $I_k(t \overset{\sigma}{\cdot} \mathcal{S}) = t \overset{\sigma}{\cdot} \mathcal{L}$ which is easy to check. The other axioms are easily verified. □

Note also that $\mathcal{K}$ acts on the splittings of Φ : If σ is a splitting, so is $I_k \circ \sigma,\ k \in \mathcal{K}$.

Proposition 5. *Let σ, ξ be two splittings of the sequence Φ with $\sigma = I_k \circ \xi$. Then $id : H^2_\sigma(\mathcal{G}, \mathcal{V}, \Phi) \to H^2_\xi(\mathcal{G}, \mathcal{V}, \Phi)$ is a linear isomorphism.*

Proof. We will use (6). Let $s \in \mathcal{S},\ s' \in \mathcal{S}'$ and $\pi(s) = \pi(s') = g$. Thus $s = x\sigma(g) = y\xi(g)$ and $s' = x'\sigma(g) = y'\xi(g)$, for some $x, x', y, y' \in \mathcal{K}$. Suppose $\sigma = I_k \circ \xi$. Then

$$\begin{aligned} kxx'\sigma(g)k^{-1} &= kxy'\xi(g)k^{-1} = xy'k\xi(g)k^{-1} \\ &= xy'(I_k \circ \xi)(g) = y'x\sigma(g) = yy'\xi(g). \end{aligned}$$

Therefore

$$\begin{aligned} I_k(\mathcal{S} \overset{\sigma}{\oplus} \mathcal{S}') &= \mathcal{S} \overset{\xi}{\oplus} \mathcal{S}' \\ \Leftrightarrow \quad [\mathcal{S} \overset{\sigma}{\oplus} \mathcal{S}'] &= [\mathcal{S} \overset{\xi}{\oplus} \mathcal{S}'] \\ \Leftrightarrow \quad [\mathcal{S}] \overset{\sigma}{\oplus} [\mathcal{S}'] &= [\mathcal{S}] \overset{\xi}{\oplus} [\mathcal{S}']. \end{aligned}$$

Thus, identity map is a group isomorphism. Since $k(tx)\xi(g)k^{-1} = (tx)k\xi(g)k^{-1} = (tx)\sigma(g)$, we have $[t \overset{\sigma}{\cdot} \mathcal{S}] = [t \overset{\xi}{\cdot} \mathcal{S}]$, so it is also linear. □

We now make our second assumption.

A2. For any two splittings ξ, σ of Φ, $I_k \circ \xi = \sigma$ for some $k \in \mathcal{K}$, i.e. $H^1(\mathcal{G}_1, \mathcal{K}) = 0$.

Definition 6. $H^2(\mathcal{G}, \mathcal{V}, \Phi) \doteq H^2_\sigma(\mathcal{G}, \mathcal{V}, \Phi)$ *for some splitting σ of Φ.*

Note that $H^2(\mathcal{G}_1, \mathcal{K}, \Phi) = 0$ by definition. We will single out the following simple fact as a proposition which will be crucial later.

Proposition 7. *Let $\mathcal{S}$ be an extension as in (4). Then the following are equivalent.*

(i) $[\mathcal{S}] = 0$ in $H^2(\mathcal{G}, \mathcal{V}, \Phi)$.

(ii) There exists some splitting σ such that $\sigma|_{\mathcal{S}}$ splits the upper sequence of (4).

(iii) There exists some splitting σ such that $\sigma(\mathcal{G}) \subset \mathcal{S}$.

Proof. $(i) \Rightarrow (ii)$: We choose some ξ. Since $[\mathcal{S}] = [\overset{\xi}{\mathcal{E}}]$ in $H^2_\xi(\mathcal{G}, \mathcal{V}, \Phi)$, we have $I_k(\overset{\xi}{\mathcal{E}}) = \mathcal{S}$ for some $k \in \mathcal{K}$. It follows that for $\sigma = k\xi k^{-1}$, $\overset{\sigma}{\mathcal{E}} = k(\overset{\xi}{\mathcal{E}})k^{-1} = \mathcal{S}$.

$(ii) \Rightarrow (iii)$: Obvious.

$(iii) \Rightarrow (i)$: Let $s \in \mathcal{S}$ with $\pi(s) = g$. Then $s = k\sigma(g)$, $k \in \mathcal{K}$. Since $\sigma(g) \in \mathcal{S}$, k is in $\mathcal{S} \cap \mathcal{K} = \mathcal{V}$. Therefore $\mathcal{S} = \overset{\sigma}{\mathcal{E}}$ so that $[\mathcal{S}] = 0$. □

Let $H^2(\mathcal{G}, \mathcal{V})$ denote the group of arbitrary extensions of $\mathcal{G}$ by $\mathcal{V}$. Thus we have the obvious homomorphism

$$\mu : H^2(\mathcal{G}, \mathcal{V}, \Phi) \longrightarrow H^2(\mathcal{G}, \mathcal{V})$$

induced by inclusion.

The space $H^2(\mathcal{G}, \mathcal{V})$ is studied for Lie groups in detail in [Hoch51]. In a similar way, the above construction of $H^2(\mathcal{G}, \mathcal{V}, \Phi)$ works through if we assume that the groups occurring in Φ are Lie groups and the groups in the upper sequence in (4) are Lie subgroups. In this case, we need to assume that all splittings are analytic. The framework in the next two sections will be a particular instance of this more general situation. We should note here that the use of analytic factor sets is not feasible in defining $H^2(\mathcal{G}, \mathcal{V})$ in general, unless $\mathcal{G}$ is simply connected (see [Hoch51], pg.105). However, this crucial point will be irrelevant in the particular situation of the next section.

3. Construction of $H^2(\mathcal{G}, \mathcal{V}, \Phi(n))$ for the second order jet group

Let $GL_2(n,\mathbb{R}) = \mathcal{G}_2(n)$ be the set of 2-jets all local diffeomorphisms of $\mathbb{R}^n$ with source and target at the origin $0 \in \mathbb{R}^n$. $\mathcal{G}_2(n)$ is a group with the operation induced by the composition of jets: For $j^2(f)_0, j^2(g)_0 \in \mathcal{G}_2(n)$ we define $j^2(f)_0 \circ j^2(g)_0 \doteq j^2(f \circ g)_0$. The same construction with 2-jets replaced by 1-jets gives the group $GL_1(n,\mathbb{R}) = \mathcal{G}_1(n)$ and a surjective homomorphism of Lie groups with kernel $\mathcal{K}(n)$ which is a vector space. Thus we have the exact sequence

$$\Phi(n): \quad 0 \longrightarrow \mathcal{K}(n) \xrightarrow{i} \mathcal{G}_2(n) \xrightarrow{\pi} \mathcal{G}_1(n) \longrightarrow 1 \tag{9}$$

In the coordinates (x^i) of $\mathbb{R}^n$, an element of $\mathcal{G}_2(n)$ is expressed in the form (f^i_j, f^i_{jk}), $1 \leqslant i,j,k \leqslant n$, and the group operation is given by

$$(f^i_j, f^i_{jk})(g^i_j, g^i_{jk}) = (f^i_a g^a_j, f^i_a g^a_{jk} + f^i_{ab} g^a_j g^b_k) \tag{10}$$

which is the chain rule of differentiation. The identity is $(\delta^i_j, 0)$ and π is given by $\pi(f^i_j, f^i_{jk}) = f^i_j$.We will denote (f^i_j, f^i_{jk}) also by (f,F) and identity by $(\mathbb{1},0)$. With this short notation, we will write the group operation as $(f,F)(g,G) = (fg, fG + Fg)$. For instance, $(f,F)^{-1} = (f^{-1}, -f^{-1}Ff^{-1})$ using our short notation. Let $(x) \to (y)$ be a coordinate change with $(g,G) \doteq (\frac{\partial y^i}{\partial x^j}(0), \frac{\partial^2 y^i}{\partial x^i \partial x^j}(0))$. We find $(g,G)(f,F)(g,G)^{-1} = (\overset{y}{f}, \overset{y}{F})$ where $(\overset{y}{f}, \overset{y}{F})$ denotes the components of (f,F) in (y^i) coordinates. We will use this superscript convention also for other objects to be defined below. Note that f in (f,F) has an invariant meaning but F alone does not.

An important property of the extension (9) is that it splits. There is the canonical splitting $s : \mathcal{G}_1(n) \to \mathcal{G}_2(n)$ defined by $s(f) = (f,0)$. Thus, the exact sequence $\Phi(n)$ satisfies the hypothesis **A1** of Section 2, and $(f,F) = ks(g)$ with $k = (\mathbb{1}, Ff^{-1}) \in \mathcal{K}(n)$. Observe the double meaning of the symbol (f,F) here and in Section 2. Nevertheless $\mathcal{G}_2(n)$ with the operation above is isomorphic to the set $\mathcal{G}_2(n)$ equipped with the operation as in Section 2 where the isomorphism is given by $(f,F) \mapsto (f, Ff^{-1})$. From now on, we will be using the symbol (f,F) in the sense of previous paragraph.

With this setting, we have

$$\begin{aligned}
(g,G)(f,0)(g,G)^{-1} &= (gf, Gf)(g^{-1}, -g^{-1}Gg^{-1}) \\
&= (gfg^{-1}, -gfg^{-1}Gg^{-1} + Gg^{-1}gfg^{-1}) \\
&= (\overset{y}{f}, \Gamma(\overset{y}{f}) - (\overset{y}{f})\Gamma), \quad \Gamma \doteq Gg^{-1}
\end{aligned}$$

and therefore

$$\begin{aligned}
\overset{y}{s}(\overset{y}{f}) &= (\overset{y}{f}, \Gamma(\overset{y}{f}) - (\overset{y}{f})\Gamma) \\
&= (\mathbb{1}, \Gamma)(\overset{y}{f}, 0)(\mathbb{1}, \Gamma)^{-1}
\end{aligned} \tag{11}$$

and not $\overset{y}{s}(\overset{y}{f}) = (\overset{y}{f}, 0)$. On the other hand, the assignment $(\overset{y}{f}) \to (\overset{y}{f}, 0)$ is clearly another splitting of (9). To avoid confusion, we will denote this splitting by $\overset{(y)}{s}$. Thus (11) becomes now

$$\overset{y}{s} = I_\Gamma \circ \overset{(y)}{s}$$

where I_Γ is the inner automorphism of $\mathcal{G}_2(n)$ defined by $\Gamma \in \mathcal{K}(n)$.

We will now show that $\Phi(n)$ satisfies also **A2**.

Proposition 8. *(i) Let σ be an arbitrary (not necessarily continuous) splitting of (9). Then there exists some coordinates (y^i) such that $\sigma = \overset{(y)}{s}$.*

(ii) Let σ_1, σ_2 be two splittings of (9). Then there exists a unique $\Gamma \in \mathcal{K}(n)$ such that $\sigma_1 = I_\Gamma \circ \sigma_2$.

Proof. Let $\sigma(f)=(f,\Omega(f))$ for some function Ω. Since σ is a homomorphism, we have $\sigma(fg)=(fg,\Omega(fg))=\sigma(f)\sigma(g)=(f,\Omega(f))(g,\Omega(g))=(fg,\Omega(f)g+f\Omega(g))$ and therefore

$$\Omega(fg)=\Omega(f)g+f\Omega(g)\quad f,g\in\mathcal{G}_1(n).$$

Setting $g=a\mathbb{1}$ for some real number $a\neq 1$, $\Omega(fg)=\Omega(gf)$ holds. Using the relation above, this yields

$$\Omega(g)f-f\Omega(g)=(a^2-a)\Omega(f) \tag{12}$$

since $\Omega(f)g=a^2\Omega(f)$ and $g\Omega(f)=a\Omega(f)$ via the groups operation defined by (10). Thus, $\Omega(f)=\Gamma f-f\Gamma$ where $\Gamma=\frac{1}{a^2-a}\Omega(g)$. Consequently we get

$$\begin{aligned}\sigma(f) &= (f,\Gamma f-f\Gamma)\\ &= (\mathbb{1},\Gamma)(f,0)(\mathbb{1},\Gamma)^{-1}\\ &= I_\Gamma\circ s \qquad f\in\mathcal{G}_1(n).\end{aligned} \tag{13}$$

Γ in (13) is unique. Indeed, if $\Gamma_1 f-f\Gamma_1=\Gamma_2 f-f\Gamma_2$ for all f, we get $(\Gamma_1-\Gamma_2)f=f(\Gamma_1-\Gamma_2)$. Setting $f=b\mathbb{1}$, yields $b^2(\Gamma_1-\Gamma_2)=b(\Gamma_1-\Gamma_2)$ which proves $\Gamma_1=\Gamma_2$ since b is arbitrary. Both claims follow immediately. □

Corollary 9. *Let $\mathcal{V}\subset\mathcal{K}(n)$ be a subspace, $\mathcal{G}\subset\mathcal{G}_1(n)$ be a Lie subgroup and suppose that $\mathcal{V}$ is a kernel for $\mathcal{G}$ as in Definition 1. Then $\Phi(n)$satisfies* **A1** *and* **A2** *and consequently the vector space $H^2(\mathcal{G},\mathcal{V},\Phi(n))$ is defined.*

Let Γ^i_{jk} denote the components of a first order symmetric connection Γ. Consider the system of geodesics

$$\ddot{x}^i+\Gamma^i_{ab}\dot{x}^a\dot{x}^b=0 \tag{14}$$

where we assume that the values of the quantities in (14) are assigned only at $0\in\mathbb{R}^n$. Let $\widetilde{\Gamma}\subset\mathcal{G}_2(n)$ be the subgroup which stabilizes (14). If $(f,F)\in\widetilde{\Gamma}$, an easy computation shows that $F=-(\Gamma f-f\Gamma)$, $f\in\mathcal{G}_1(n)$. Comparing with (13), we see that there is a 1-1 correspondence between first order symmetric connections at $0\in\mathbb{R}^n$ and splittings of (9), which justifies our notation Γ. Our examples in Section 5 will show the impossibility of studying certain second order geometric structures using only Γ even if we prolong Γ to a second order connection.

Based on our computations in search for a nontrivial $H^2(\mathcal{G},\mathcal{V},\Phi(n))$, we are led to the following conjecture:

C1 : Let T denote the tensor space of $\mathbb{R}^n$ and consider the linear representation of $\mathcal{G}_2(n)$ on $J^1(T)$. Suppose that $\mathcal{S}$ is the stabilizer of some $\phi_1,\phi_2,...,\phi_k\in J^1(T)$. Then $H^2(\pi(\mathcal{S}),\mathcal{V},\Phi(n))=0$.

However, linear representations of $\mathcal{G}_2(n)$ are much richer than those in **C1** according to [Ter78]. We believe that **C1** holds also for k different actions (not necessarily linear) of $\mathcal{G}_2(n)$ if these actions are prolonged actions of $\mathcal{G}_1(n)$. For instance, in Example 3 of Section 5, $\mathcal{S}$ will be the stabilizer of some $\Lambda\in\mathcal{G}_2(n)$ where $\mathcal{G}_2(n)$ acts on itself by inner automorphisms, i.e. $\mathcal{S}$ will be the centralizer of some $\Lambda\in\mathcal{G}_2(n)$. However, an arbitrary Λ will not serve our purpose.

4. The homomorphism $H^2(\mathcal{G},\mathcal{V},\Phi(n))\to H^2(\mathcal{L}\mathcal{G},\mathcal{V},\mathcal{L}\Phi(n))$

Consider the Lie algebra exact sequence

$$\mathcal{L}\Phi(n):\quad 0\longrightarrow\mathcal{K}(n)\xrightarrow{i}\mathcal{L}\mathcal{G}_2(n)\xrightarrow{\pi}\mathcal{L}\mathcal{G}_1(n)\longrightarrow 1 \tag{15}$$

where $(\mathbb{1},\Gamma)\in\mathcal{K}(n)$ is identified with $(0,\Gamma)\in\mathcal{L}\mathcal{K}(n)$. We will denote $(0,\Gamma)$ simply by Γ.

An element of $\mathcal{LG}_2(n)$ is of the form (g^i_j, G^i_{jk}) and the bracket is

$$[(g^i_j, G^i_{jk}), (h^i_j, H^i_{jk})] = (g^i_a h^a_j - h^i_a g^a_j,\ g^i_a H^a_{jk} + G^i_{ak} h^a_j + G^i_{aj} h^a_k \\ - h^i_a G^a_{jk} - H^i_{ak} g^a_j - H^i_{aj} g^a_k) \tag{16}$$

Recall that the action of $f \in \mathcal{G}_1(n)$ on $\mathcal{K}(n)$ is given by $\Gamma^i_{jk} \to f^i_a \Gamma^a_{bc} (f^{-1})^b_j (f^{-1})^c_k$. Differentiating at the identity, we obtain a map $\mathcal{LG}_1(n) \to \mathrm{End}(\mathcal{K}(n))$ which maps $g \in \mathcal{LG}_1(n)$ to $\overline{g}$ defined by

$$\overline{g}(\Gamma^i_{jk}) = g^i_a \Gamma^a_{jk} - \Gamma^i_{ak} g^a_j - \Gamma^i_{aj} g^a_k.$$

It follows that

$$\overline{[g,h]}_{\mathcal{LG}_1(n)} = \overline{g} \circ \overline{h} - \overline{h} \circ \overline{g} = [\overline{g}, \overline{h}]_{gl(\mathcal{K})}, \qquad g, h \in \mathcal{LG}_1(n)$$

and thus we have a representation $\mathcal{LG}_1(n) \to gl(\mathcal{K}(n))$. Now (16) becomes

$$[(g,G),(h,H)]_{\mathcal{LG}_2(n)} = ([g,h]_{\mathcal{LG}_1(n)}, \overline{g}(H) - \overline{h}(G)) \tag{17}$$

We will now define the Lie algebra analogue $H^2(\mathfrak{g}, \mathcal{V}, \mathcal{L}\Phi(n))$ of $H^2(\mathcal{G}, \mathcal{V}, \Phi(n))$. In view of Sections 2, 3, this construction is now quite straightforward and we will omit the details.

$\mathcal{L}\Phi(n)$ splits and $\mathcal{K}(n)$ acts transitively (by adjoint action) on all splittings of (15). We define kernels and restricted Lie algebra extensions as before. Such extensions are closed with respect to Baer sum. Passing over to the equivalence classes modulo the adjoint action of $\mathcal{K}(n)$, we arrive at the vector space $H^2(\mathfrak{g}, \mathcal{V}, \mathcal{L}\Phi(n))$. We obtain the well defined mapping of sets $H^2(\mathcal{G}, \mathcal{V}, \Phi(n)) \to H^2(\mathcal{LG}, \mathcal{V}, \mathcal{L}\Phi(n))$ which is a homomorphism by construction. Thus we obtain the diagram

$$\begin{array}{ccc} H^2(\mathcal{G}, \mathcal{V}, \Phi(n)) & \longrightarrow & H^2(\mathcal{LG}, \mathcal{V}, \mathcal{L}\Phi(n)) \\ \downarrow & & \downarrow \\ H^2(\mathcal{G}, \mathcal{V}) & \longrightarrow & H^2(\mathcal{LG}, \mathcal{V}) \end{array} \tag{18}$$

where the vertical homomorphisms are induced by inclusions. The cohomology groups $H^*(\mathfrak{g}, \mathcal{V})$ are defined in [CE48] for an arbitrary Lie algebra $\mathfrak{g}$ and representation $\mathcal{V}$. See [HM62] for $H^*(\mathcal{G}, \mathcal{V})$ (again in the general case). It is proved in [Hoch51] that the lower map in (18) is an isomorphism if $\mathcal{G}$ is simply connected. This isomorphism is generalized to $H^*(\mathcal{G}, \mathcal{V}) \to H^*(\mathcal{LG}, \mathcal{V})$ in [VEst53] with the assumption that $\mathcal{G}$ is homologically trivial in certain dimensions.

Q1 : Show that the upper homomorphism in (18) is surjective and prove a Van Est type theorem for this homomorphism.

Our examples in the next section will indicate that, if such a theorem exists, it may not be possible to formulate its hypothesis using the topology of $\mathcal{G}$ only, as one may need deeper properties of $\mathcal{G}$.

Q2 : Are the vertical homomorphisms in (18) surjective?

Now suppose $[\mathcal{S}] = 0$ in $H^2(\mathcal{G}, \mathcal{V}, \Phi(n))$. In view of Proposition 7 and Diagram (7), there exists a splitting σ of $\Phi(n)$ which splits also the upper sequence in (1). Equation (13) shows that σ is of the form $\sigma(f) = (f, \Gamma f - f\Gamma) = (f, F)$ or

$$F^i_{jk} = \Gamma^i_{ab} f^a_j f^b_k - f^i_a \Gamma^a_{jk}, \quad f \in \mathcal{G}_1(n) \tag{19}$$

for some constants Γ^i_{jk}. Similarly $[\mathcal{LS}] = 0$ in $H^2(\mathcal{LG}, \mathcal{V}, \mathcal{L}\Phi(n))$ iff there exists a splitting σ of $\mathcal{L}\Phi(n)$ whose restriction to $\mathcal{LG}$ splits $\mathcal{LS}$. In this case, $\sigma(g) = (g, G(g))$ where $G(g)$ is defined by

$$G(g)^i_{jk} = \Gamma^i_{ak} g^a_j + \Gamma^i_{aj} g^a_k - g^i_a \Gamma^a_{jk}, \quad g \in \mathcal{LG}_1(n) \tag{20}$$

or shortly $\sigma(g) = (g, -\overline{g}(\Gamma))$ for some $\Gamma \in \mathcal{K}(n)$. This is obtained by differentiating (19) at the identity $(\mathbb{1}, 0)$ and follows also from (16) by setting $(h^i_j, H^i_{jk}) = (0, \Gamma^i_{jk})$.

5. Nontriviality of $H^2(\mathcal{G},\mathcal{V},\Phi(n))$

In this section we will present three nontrivial examples.

Example 1 : This extreme example will show that $H^2(\mathcal{G},\mathcal{V},\Phi(n))$ can be nontrivial even if $\mathcal{V}=0$. We choose $\lambda\in\mathcal{LG}_1(n)$ satisfying i) $\overline{\lambda}\neq 0$; ii) $R(\overline{\lambda})\subsetneq\mathcal{K}(n)$ where $R(\overline{\lambda})$ is the range of $\overline{\lambda}$. Note that no λ satisfies (i) and (ii) for $n=1$, so we assume $n\geqslant 2$. We choose $\theta\notin R(\overline{\lambda})$ and define $\mathfrak{g}\doteq\operatorname{span}\{\lambda\}\subset\mathcal{LG}_1(n)$ and $\mathfrak{s}\doteq\operatorname{span}\{(\lambda,\theta)\}\subset\mathcal{LG}_2(n)$. We define $\pi:\mathfrak{s}\to\mathfrak{g}$ by $\pi(\lambda,\theta)=\lambda$ obtaining the restricted extension of abelian Lie algebras

$$0\longrightarrow\mathcal{V}\longrightarrow\mathfrak{s}\stackrel{\pi}{\longrightarrow}\mathfrak{g}\longrightarrow 0 \tag{21}$$

where $\mathcal{V}=0$! Suppose $[\mathfrak{s}]=0$ in $H^2(\mathfrak{g},\mathcal{V},\mathcal{L}\Phi(n))$. Then there exists some $K_0\in\mathcal{K}$ such that $(\lambda,\overline{\lambda}(K_0))=(\lambda,\theta)$ and therefore $\theta\in R(\overline{\lambda})$, contradicting our choice. It is easy to see that $\operatorname{span}\{\lambda,\theta_1\}$ and $\operatorname{span}\{\lambda,\theta_2\}$ determine the same class in $H^2(\mathfrak{g},\mathcal{V},\mathcal{L}\Phi(n))$ iff $\theta_1-\theta_2\in R(\overline{\lambda})$ and therefore $\dim H^2(\mathfrak{g},\mathcal{V},\mathcal{L}\Phi(n))=\dim\mathcal{K}-\dim R(\overline{\lambda})$. In particular, we see that the 1-parameter subgroup $e^{t(\lambda,\theta)}$ does not split over $e^{t\lambda}$.

Example 2 : We will give an example where $H^2(\mathfrak{g},\mathcal{V},\mathcal{L}\Phi(n))$ is nonzero. This would imply the nontriviality of $H^2(\mathcal{G},\mathcal{V},\Phi(n))$ if the upper homomorphism in (18) is surjective as claimed in **Q1**. Supposing $n\geqslant 2$, we choose $\lambda\in\mathcal{LG}_1(n)$ satisfying $\overline{\lambda}\neq 0$ but $\overline{\lambda}^2=0$. We also choose $\theta\in\mathcal{K}(n)$ which is not in $N(\overline{\lambda})$, the nullspace of $\overline{\lambda}$. Let us define $\mathcal{C}(\lambda,\theta)\doteq\{(x,X)\in\mathcal{LG}_2(n)\mid[(x,X),(\lambda,\theta)]=0\}$. By (17), we have

$$\begin{aligned}[x,\lambda]_{\mathcal{LG}_1(n)}&=0\\ \overline{x}(\theta)-\overline{\lambda}(X)&=0\end{aligned} \tag{22}$$

Consider $\mathfrak{g}\doteq\pi(\mathcal{C}(\lambda,\theta))$ in $\mathfrak{C}(\lambda)\doteq\{y\in\mathcal{LG}_1(n)\mid[y,\lambda]_{\mathcal{LG}_1(n)}=0\}$. Since $\lambda\in\mathfrak{g}$, we have $1\leqslant\dim\mathfrak{g}\leqslant\dim\mathfrak{C}(\lambda)$. By (22), the nullspace $N(\pi)$ of $\pi:\mathcal{C}(\lambda,\theta)\to\mathfrak{g}$ is $(0,N(\overline{\lambda}))\cong N(\overline{\lambda})$. Thus we obtain the restricted Lie algebra extension

$$0\longrightarrow N(\overline{\lambda})\longrightarrow\mathcal{C}(\lambda,\theta)\longrightarrow\mathfrak{g}\longrightarrow 0 \tag{23}$$

Now suppose $[\mathcal{C}(\lambda,\theta)]=0$ in $H^2(\mathfrak{g},N(\overline{\lambda}),\mathcal{L}\Phi(n))$. Then there would exist some $K_0\in\mathcal{K}(n)$ such that $(x,\overline{x}(K_0))\subset\mathcal{C}(\lambda,\theta)$ for all $x\in\mathfrak{g}$. Now (22) gives $\overline{x}(\theta)-(\overline{\lambda}\circ\overline{x})(K_0)=0$. Setting $x=\lambda$, we obtain $\overline{\lambda}(\theta)=0$ since $\overline{\lambda}^2=0$, which contradicts our choice $\theta\notin N(\overline{\lambda})$.

Example 3 : In this example we will directly construct the group $\mathcal{S}$ for $n=2$. We do not know whether this example is a special case of Example 2. Let $\Lambda=(\lambda,\theta)\in\mathcal{G}_2(n)$ and consider the subgroup $\mathcal{S}\subset\mathcal{G}_2(n)$ consisting of elements in $\mathcal{G}_2(n)$ which centralize (commute with) Λ, i.e. (f,F) is in $\mathcal{S}$ iff $(\lambda,\theta)(f,F)=(f,F)(\lambda,\theta)$, that is, $(\lambda f,\lambda F+\theta f)=(f\lambda,f\theta+F\lambda)$, or equivalently

$$\lambda f=f\lambda \tag{24}$$

$$\theta f-f\theta=F\lambda-\lambda F \tag{25}$$

In coordinates, equations (24) and (25) become

$$\lambda^i_a f^a_j=f^i_a\lambda^a_j \tag{26}$$

$$\theta^i_{ab}f^a_jf^b_k-f^i_a\theta^a_{jk}=F^i_{ab}\lambda^a_j\lambda^b_k-\lambda^i_aF^a_{jk} \tag{27}$$

Equations (26) and (27) are the defining equations of $\mathcal{S}$. With an abuse of notation, we will write $\mathcal{S}=(f,F)$.

We choose

$$\lambda=\begin{bmatrix}1&c\\0&1\end{bmatrix},\quad c\neq 0. \tag{28}$$

We fix λ once and for all. We also choose

$$\theta^1_{11} \neq -\theta^2_{12} \quad \text{or} \quad \theta^2_{11} \neq 0. \tag{29}$$

Now (24) defines the group

$$\mathcal{P} = \{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \mid x, y \in \mathbb{R}, x \neq 0\} \subset \mathcal{G}_1. \tag{30}$$

Let $\mathcal{G} = \pi(\mathcal{S})$. Thus, $\mathcal{G} \subset \mathcal{P}$. Substituting (30) into (25), we define the function $\Xi(f, F, \lambda, \theta) \doteq \theta f - f\theta - F\lambda + \lambda F$ which depends on 7 parameters (λ, θ) and 8 variables (f, F) and thus $\Xi : \mathbb{R}^8 \to \mathbb{R}^6$. Now Ξ vanishes for $(f, F) = (\mathbb{1}, 0)$ and (29) implies that Ξ has constant rank 5 near $(\mathbb{1}, 0)$. Thus (26) and (27) define a smooth manifold of dimension 3 near $(\mathbb{1}, 0)$ for any choice of parameters satisfying (29). The details of the verification will be evident below.

To find the equations for $\mathcal{LS}$, we differentiate (26) and (27) at the identity $(\mathbb{1}, 0)$, which gives

$$\lambda^i_a g^a_j = g^i_a \lambda^a_j \tag{31}$$

$$\theta^i_{ak} g^a_j + \theta^i_{ja} g^a_k - g^i_a \theta^a_{jk} = G^i_{ab} \lambda^a_j \lambda^b_k - \lambda^i_a G^a_{jk} \tag{32}$$

where $\mathcal{LS} = (g, G)$. Now (31) defines

$$\mathcal{LP} = \{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \mid x, y \in \mathbb{R}\} \subset \mathcal{LG}_1. \tag{33}$$

We have 6 equations in (32) for the choices $(i, j, k) = (1, 1, 1)$, $(1, 2, 1)$, $(1, 2, 2)$, $(2, 1, 1)$, $(2, 2, 1)$, $(2, 2, 2)$. Substituting $g^1_1 = g^2_2 = x$, $g^1_2 = y$, $g^2_1 = 0$ into the LHS of (32) and $\lambda^1_1 = \lambda^2_2 = 1$, $\lambda^1_2 = c$, $\lambda^2_1 = 0$ into the RHS of (32), adding the first equation to fifth equation and simplifying, we find

$$-\theta^1_{11} x + \theta^2_{11} y = cG^2_{11} \tag{34}$$

$$-\theta^1_{12} x + (\theta^2_{12} - \theta^1_{11}) y = c(G^2_{12} - G^1_{11}) \tag{35}$$

$$-\theta^1_{22} x + (\theta^2_{22} - 2\theta^1_{12}) y = cG^2_{22} - c^2 G^1_{11} - 2cG^1_{12} \tag{36}$$

$$-\theta^2_{11} x = 0 \tag{37}$$

$$(-\theta^1_{11} - \theta^2_{12}) x = 0 \tag{38}$$

$$-\theta^2_{22} x - 2\theta^2_{12} y = -c^2 G^2_{11} - 2cG^2_{12} \tag{39}$$

We observe that the linear forms on the RHS of (34), (35), (36), (39) are linearly independent. Setting $x = y = 0$ in (34)-(39), we get the equations which define $\mathcal{V}$. Thus $\dim \mathcal{V} = 2$ as G^1_{22} and G^1_{12} are free variables. Note that $\mathcal{V}$ depends only on λ.

Let $\mathfrak{g} = \mathcal{LG} = \mathcal{L}\pi(\mathcal{S}) = \pi(\mathcal{LS}) \subset \mathcal{LP}$. Now assume $\mathfrak{g} = \mathcal{LP}$. This means that (34)-(39) have solutions for the 6 unknowns $G^2_{11}, \ldots, G^2_{12}$ for all $x, y \in \mathbb{R}$, which is impossible in view of (29) and (37), (38). Thus we must have $x = 0$ and $\mathfrak{g}$ is contained in the Lie subalgebra of $\mathcal{LP}$ obtained by setting $x = 0$ in (33). Now setting $x = 0$ on the LHS of (34)-(39) and letting y be arbitrary, (34)-(39) have solutions for all y and therefore $\mathfrak{g}$ also contains this subalgebra of $\mathcal{LP}$. Thus, we conclude

$$\mathfrak{g} = \{ \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \mid \ y \in \mathbb{R}\}. \tag{40}$$

Now for any choice of parameters λ^i_j satisfying (29), we constructed a restricted Lie algebra extension as in (21) with $\mathfrak{s} = \mathcal{LS}$, $\dim \mathcal{V} = 2$ and $\dim \mathfrak{g} = 1$. However, note the possibility that we may have other restricted extensions which project onto $\mathfrak{g}$ with kernel $\mathcal{V}$ but are not equivalent to those defined in this way.

DEFINITION 10. *Let $\mathcal{S}$ be a restricted extension as in (4). $\mathcal{S}$ is called a centralizer if it centralizes some $\Lambda \in \mathcal{G}_2(n)$ as above. In this case, the restricted Lie algebra extension $\mathcal{LS}$ is also called a centralizer.*

Definition 11. *A centralizer* $\mathcal{LS}$ *is said to be*

of type I if $\theta^1_{11} = -\theta^2_{12}$ *and* $\theta^2_{11} = 0$,
of type II if $\theta^1_{11} = -\theta^2_{12}$ *and* $\theta^2_{11} \neq 0$,
of type III if $\theta^1_{11} \neq -\theta^2_{12}$ *and* $\theta^2_{11} = 0$.

We should be careful with the definition of type I: as we have seen above, the defining conditions for centralizers of type II and III impose the same condition on (30) and force the projection to be $\mathfrak{g}$ in (40), whereas a centralizer of type I imposes no conditions on (30). Therefore, by a centralizer $\mathcal{LS}$ of type I we mean the solutions (34)-(39) once we set $x = 0$ on LHS of these equations. In the invariant language, we construct first the split extension $\mathcal{E}$ with projection (30) and define $\mathcal{LS}$ by the elements in $\mathcal{E}$ which project onto $\mathfrak{g}$.

For λ fixed as in (28) (so for the corresponding fixed $\mathcal{V}$) we make

Definition 12. $H^2_C(\mathfrak{g}, \mathcal{V}, \mathcal{L}\Phi(n)) \doteq \{x \in H^2(\mathfrak{g}, \mathcal{V}, \mathcal{L}\Phi(n)) \mid x = [\mathcal{LS}]$ *for some centralizer* $\mathcal{LS}\}$.

Lemma 13. *(i)* $H^2_C(\mathfrak{g}, \mathcal{V}, \mathcal{L}\Phi(n))$ *is a subspace of* $H^2(\mathfrak{g}, \mathcal{V}, \mathcal{L}\Phi(n)$.

(ii) Let $\alpha \in H^2_C(\mathfrak{g}, \mathcal{V}, \mathcal{L}\Phi(n))$. *Then* $\alpha = 0$ *if and only if all representatives of* α *are of type I.*

(iii) Let $\alpha, \beta \in H^2_C(\mathfrak{g}, \mathcal{V}, \mathcal{L}\Phi(n))$ *both have representatives of type II (or III). Then* α, β *are linearly dependent.*

(iv) α, β *have representatives of types II and III, respectively if and only if* $\{\alpha, \beta\}$ *is a basis of* $H^2_C(\mathfrak{g}, \mathcal{V}, \mathcal{L}\Phi(n))$.

Proof. (i) Let $\mathcal{LS}_1 = (g, G_1)$ and $\mathcal{LS}_2 = (g, G_2)$ centralize $\Lambda_1 = (\lambda, \theta)$, $\Lambda_2 = (\lambda, \vartheta)$ respectively and let $a, b \in \mathbb{R}$. Now $a[\mathcal{LS}_1] + b[\mathcal{LS}_2] = [(g, aG_1 + bG_2)]$ which centralizes $(\lambda, a\theta + b\vartheta)$.

(ii) Suppose that (34)-(39) are the defining equations of a representative for α. We substitute $g^1_1 = g^2_2 = g^2_1 = 0$, $g^1_2 = y$, into RHS of (20) and then substitute (20) into RHS of (34)-(39). Now (34), (35), (36), (39) give

$$\begin{array}{rcl} \theta^2_{11} y &=& 0 \\ (\theta^2_{12} - \theta^1_{11}) y &=& 2c\Gamma^2_{11} y \\ (\theta^2_{22} - 2\theta^1_{12}) y &=& (4c\Gamma^2_{12} + c^2\Gamma^2_{11} - 2c\Gamma^1_{11}) y \\ 2\theta^2_{12} y &=& 2c\Gamma^2_{11} y \end{array} \tag{41}$$

The class $\alpha = 0$ if and only if the system (41) has a solution for $\Gamma^2_{11}, \Gamma^2_{12}, \Gamma^1_{11}$, Γ^2_{11} for all y which implies $\theta^2_{11} = 0$ and $\theta^1_{11} = -\theta^2_{12}$.

(iii) Let $\alpha = [\mathcal{LS}_1]$, $\beta = [\mathcal{LS}_2]$ where $\mathcal{LS}_1$, $\mathcal{LS}_2$ centralize $\Lambda_1 = (\lambda, \theta)$, $\Lambda_2 = (\lambda, \vartheta)$ respectively. Suppose that $\mathcal{LS}_1$, $\mathcal{LS}_2$ are of type II so that $\theta^1_{11} = -\theta^2_{12}$, $\theta^2_{11} \neq 0$ and $\vartheta^1_{11} = -\vartheta^2_{12}$, $\vartheta^2_{11} \neq 0$. Choose $a, b \in \mathbb{R}$ both nonzero satisfying $a\theta^2_{11} + b\vartheta^2_{11} = 0$ so that $a\alpha + b\beta = 0$. Similar argument holds if they are both of type III.

(iv) Follows now easily. □

Lemma 14. *Let* $\mathfrak{s}$ *be any restricted Lie algebra extension of* $\mathfrak{g}$ *by* $\mathcal{V}$. *Then*

(i) $\mathfrak{s}$ *is a centralizer, i.e.* $\mathfrak{s}$ *represents a class in* $H^2_C(\mathfrak{g}, \mathcal{V}, \mathcal{L}\Phi(n))$.

(ii) $\mathfrak{s}$ *is abelian.*

Proof. (i) Since $\dim \mathfrak{s} = 3$, $\mathfrak{s}$ has a basis of the form $\{(0, \Gamma), (0, \Gamma'), (\mathbf{y}, H)\}$ where $\{\Gamma, \Gamma'\}$ is a basis for $\mathcal{V}$ and $\mathbf{y} = \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \in \mathfrak{g}$. By definition, $(0, \Gamma)$ (or $(0, \Gamma')$), when inserted in place of (g, G), satisfies the system (32) for any θ. Now if (g, G) is replaced by $(\mathbf{y}, H)$ in (32), there is always a solution for θ. This proves that $\mathfrak{s}$ is always contained in a centralizer.

Moreover, if $\theta^2_{11} \neq 0$ or $\theta^1_{11} + \theta^2_{12} \neq 0$, then centralizer of (λ, θ) has dimension 3 therefore it is equal to $\mathfrak{s}$. If both $\theta^2_{11} = 0$ and $\theta^1_{11} + \theta^2_{12} = 0$ then $\mathfrak{s}$ is of type I.

(ii) For the basis above, $(0, \Gamma)$ commutes with $(0, \Gamma')$ while

$$[(0, \Gamma), (Y, G)] = (0; 0, 0, -y^2\Gamma^1_{11}, 0, 0, -y^2\Gamma^2_{11}).$$

Here the entries after the semicolon correspond to $(i, j, k) = (1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 2)$, $(2, 1, 1)$, $(2, 1, 2)$, $(2, 2, 2)$, respectively. Since $\Gamma \in \mathcal{V}$, it follows from the discussion following equation (39) that Γ^1_{11} and Γ^2_{11} are both 0. □

Proposition 15. *For $\mathcal{V}$ and $\mathfrak{g}$ as above, we have*

(i) $\dim H^2(\mathfrak{g}, \mathcal{V}, \mathcal{L}\Phi(n)) = 2$,

(ii) $\dim H^2(\mathfrak{g}, \mathcal{V}) = 0$.

Proof. The first statement follows from Lemma 13(iv) and Lemma 14(i). The second statement is equivalent to saying that any extension $\mathfrak{s}$ of $\mathfrak{g}$ by $\mathcal{V}$ splits. This follows from the fact that dim $\mathfrak{g} = 1$. In fact, take some $y \in \mathfrak{s} - \mathcal{V}$. Since $x = \pi(y)$ is nonzero, x spans $\mathfrak{g}$. Define a splitting σ by $\sigma(x) = y$. □

Note that (21) is simply an exact sequence of vector spaces in view of Lemma 14(ii) and is quite uninteresting if we detach it from $\mathcal{L}\Phi(n)$.

We would like to indicate the following interesting point: Let λ be a $(1,1)$-tensor defined near the origin of $\mathbb{R}^n$. Let $\mathcal{S} \subset \mathcal{G}_2(n)$ be the subgroup which stabilizes $(j^1\lambda)_0 = (\lambda^i_j, \vartheta^i_{jk})$. The defining equations of $\mathcal{S}$ are

$$\lambda^i_a f^a_j = f^i_a \lambda^a_j$$
$$\vartheta^i_{ab} f^a_j f^b_k - f^i_a \vartheta^a_{jk} = F^i_{ak}\lambda^a_j - \lambda^i_a F^a_{jk}$$

which is the same as (26) and (27) except one term. This fact suggests, but does not prove, that $\mathcal{S}$ in our example may not arise from a representation in the way as in conjecture **C1**.

It is easy to generalize the above example to $n = 3$. The computation is more involved but can be done by hand. It is possible to arrange the parameters so that the projection $\mathfrak{g}$ is a subalgebra of the strictly upper (lower) triangular matrices and $\mathfrak{g}$ can be determined in various ways depending on the conditions we impose on the parameters. We also carried out the calculations for $n = 4, 5$ on computer. However, with increasing n it becomes soon evident that it is not possible to understand the underlying structure, that is, the meaning of the conditions which cause nontriviality, by such computations even though one is easily convinced that nontriviality will persist for all n. We will postpone a conceptual study to some future work. To motivate our remarks in next section, we will content ourselves here with the following conjecture:

C2 : Let $\mathcal{U}^+(n) \subset SL(n, \mathbb{R})$ be the unipotent subgroup consisting of upper triangular matrices with $1'$s on the diagonal. Then there exists a kernel $\mathcal{V}$ for $\mathcal{U}^+(n)$ such that $H^2(\mathcal{U}^+(n), \mathcal{V}, \Phi(n))$ is nontrivial.

More generally we can ask the following question.

Q3 : For which subgroups $\mathcal{G} \subset \mathcal{G}_1(n)$ can we have $H^2(\mathcal{G}, \mathcal{V}, \Phi(n)) \neq 0$ where $\mathcal{V}$ is some kernel for $\mathcal{G}$?

Q3 is quite relevant from global point of view: Suppose $[\mathcal{S}] \neq 0$ in $H^2(\mathcal{G}, \mathcal{V}, \Phi(n))$ and $\pi(\mathcal{S}) = \mathcal{G}$. Once we have some first order $\mathcal{G}$-structure on a differentiable manifold M, there is no further topological obstruction to the existence of some second order $\mathcal{S}$-structure since the fibers of the bundle $\mathcal{G}_2(n)/\mathcal{S} \to \mathcal{G}_1(n)/\mathcal{G}$ are homeomorphic to $\mathcal{K}/\mathcal{V}$ and are thus contractible. So the information $[\mathcal{S}] \neq 0$ cannot be detected topologically. It is therefore relevant to know for which $\mathcal{G}$-structures this phenomenon can happen.

6. Some remarks

Let G be a Lie group, H a Lie subgroup and consider the action of G on the homogeneous space $M = G/H$. Now H acts on the tangent space T_oM at $o =$ coset of H with kernel $H_1 \subset H$. If $H_1 = \{e\}$, then $f \in G$ is uniquely determined as a transformation of M by its 1-jet. This is the case, for instance, if G is compact. If H_1 is nontrivial, the second order term in the Taylor expansion of f contains information in order to determine f. Iterating this process (see [Sha97], pg. 160-162) we obtain a descending chain of normal subgroups $H = H_0 \supset H_1 \supset H_2 \supset \dots$ which stabilizes at $\{e\}$ if G acts effectively on M. The smallest k where $H_k = \{e\}$ is the order of the effective Klein pair (G, H). In this case, a transformation $f \in G$ of M is uniquely determined by its k-jet. For simplicity, we will assume $k = 2$ so that we have the restricted extension

$$0 \longrightarrow H_1 \longrightarrow H \longrightarrow H/H_1 \longrightarrow 1 \tag{42}$$

We will refer to [Kob72], pg. 142, for an explicit description of the spaces in (42) in the case of projective and conformal structures which are special parabolic geometries [CSS01].

Now for $f \in G$ with $f(x) = y$, let $j^2(f)^x_y$ denote the 2-jet of f. Define $\mathcal{S}^x_y \doteq \{j^2(f)^x_y \mid f \in G, f(x) = y\}$ and $\mathcal{S} \doteq \cup_{x,y\in M}\mathcal{S}^x_y$. Now $\mathcal{S}$ is a second order transitive Lie equation on M in finite form (see [GS] and the references therein) which is a very special groupoid. We have the projection $\mathcal{S} \to \mathcal{G}$ with the obvious definition for $\mathcal{G}$. For any $x \in M$, we have the restricted extension

$$0 \longrightarrow \mathcal{V}^x_x \longrightarrow \mathcal{S}^x_x \to \mathcal{G}^x_x \longrightarrow 1 \tag{43}$$

and (43) coincides with (42) if $x = o$. Note that $\mathcal{G}$ acts on the vector bundle $\mathcal{V} \to M$ where $\mathcal{V} \doteq \cup_{x\in X}\mathcal{V}^x_x$ and we have the extension

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{S} \longrightarrow \mathcal{G} \longrightarrow 1$$

of Lie equations.

Generalizing the action of G on M, let $\mathcal{S}$ be an arbitrary second order transitive Lie equation in finite form on a differentiable manifold M with infinitesimal Lie equation $\mathcal{LS}$ [GS]. We may think $\mathcal{S}$ as a second order approximation to some ideal group G which acts transitively on M but may be far from being a Lie group. In this setting, we defined in [GOO1] the global analogues $H^2(\mathcal{G}, \mathcal{V}, M(\Theta))$ and $H^2(\mathcal{LG}, \mathcal{V}, M(\mathcal{L}\Theta))$ of $H^2(\mathcal{G}, \mathcal{V}, \Phi)$ and $H^2(\mathcal{LG}, \mathcal{V}, \mathcal{L}\Phi)$ respectively where $M(\Theta)$ ($M(\mathcal{L}\Theta)$) denotes the universal second order transitive Lie equation in finite (infinitesimal) form on M. It turns out that $H^2(\mathcal{G}, \mathcal{V}, M(\Theta))$ is isomorphic to $H^2(\mathcal{G}^x_x, \mathcal{V}^x_x, \Phi(n)_x)$ for all $x \in M$. However, the definition of $H^2(\mathcal{LG}, \mathcal{V}, M(\mathcal{L}\Theta))$ incorporates "displacement along M" through the bracket of jets of vector fields defined in terms of the Spencer operator and thus $H^2(\mathcal{LG}, \mathcal{V}, M(\mathcal{L}\Theta))$ contains, we believe, global information. For instance, $H^2(\mathcal{LG}, \mathcal{V}, M(\mathcal{L}\Theta)) = 0$ if $M = \mathbb{R}^n$ even if $H^2(\mathcal{LG}^p_p, \mathcal{V}^p_p, \mathcal{L}\Phi(n)_p) \neq 0$ as in our examples.

In this way we obtain the global analogue

$$\begin{array}{ccc} H^2(\mathcal{G}, \mathcal{V}, M(\Theta)) & \longrightarrow & H^2(\mathcal{LG}, \mathcal{V}, M(\mathcal{L}\Theta)) \\ \downarrow & & \downarrow \\ H^2_d(\mathcal{G}, \mathcal{V}) & \longrightarrow & H^2(\mathcal{LG}, \mathcal{V}) \end{array} \tag{44}$$

of (18). The cohomology groups $H^*(\mathcal{A}, V)$ are studied in detail in [Mac87] for an arbitrary algebroid $\mathcal{A}$ and representation $\mathcal{V} \to M$ (see the references in [Mac87] for original sources). The reader will find the definition of $H^*_d(\mathcal{G}, \mathcal{V})$ (for general groupoids and representations) in [Cra03] and the references therein. In view of Proposition 15, we may hope that the upper row of (44) has a richer structure than the lower row. We believe that second order connections will play a fundamental role in understanding the structure of the upper row of (44). An attempt in this direction is made in [GOO1] based on the construction of [Tel72] which in particular recaptures Chern-Weil characteristic classes if applied to the Atiyah sequence of a principle bundle (see the Appendix A of [Mac87] for a detailed account on the Atiyah sequence) and also produces characteristic classes also for Gelfand-Fuks cohomology.

As a part of our program (see [Ort06]), we define and show the nontriviality of $H^i(\mathcal{G}, \mathcal{V}, \Phi(n))$, $i = 0, 1$ in [GOO2], in which some progress is being made about the following question.

Q4 : Define a cochain complex whose i^{th} cohomology group coincides with $H^i(\mathcal{G}, \mathcal{V}, \Phi(n))$, $i = 0, 1, 2$. Using this complex, define $H^*(\mathcal{G}, \mathcal{V}, \Phi(n))$ and show the functorial properties of these groups. More generally, consider the exact sequence of jet groups

$$\Phi_{r,s}(n): \quad 0 \longrightarrow \mathcal{K}_{r,s}(n) \xrightarrow{i} \mathcal{G}_r(n) \xrightarrow{\pi} \mathcal{G}_s(n) \longrightarrow 1$$

where $r \geqslant s+1$ and the kernel $\mathcal{K}_{r,s}(n)$ is nilpotent. For restricted extensions contained in $\Phi_{r,s}(n)$, define the cohomology groups $H^*(\mathcal{G}, \mathcal{V}, \Phi_{r,s}(n))$ and derive their functorial properties.

Naturally, the next task could be formulated as follows:

Q5 : Answer **Q4** in the global situation.

References

CSS01 A. Cap, J. Slovak, V. Soucek, *Bernstein-Gelfand-Gelfand sequences*, Ann. of Math. **154** (2001), 97–113.

CE48 C. Chevalley, S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85–124.

Cra03 M. Crainic, *Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes*, Comment. Math. Helv. **78** (2003), 681–721.

VEst53 W. T. Van Est, *Group cohomology and Lie algebra cohomology in Lie groups. I*, Indagationes Math. **15** (1953), 484–492.

GS H. Goldschmidt, D. C. Spencer, *On the nonlinear cohomology of Lie equations. I,II,III,IV*, Acta Math. **136** (1976), 103–170; 171–239; J. Differential Geom. **13** (1978); 409–453, (1979), 455–526.

GOO1 B. Gürel, E. Ortaçgil, F. Öztürk, *Second order geometry*, in preparation.

GOO2 B. Gürel, E. Ortaçgil, F. Öztürk, *Jet-cohomology groups of low orders*, in preparation.

Hoch51 G. Hochshild, *Group extensions of Lie groups*, Ann. of Math. **54** (1951), 96–109.

HM62 G. Hochschild, G. D. Mostow, *Cohomology of Lie groups*, Illinois J. Math. **6** (1962), 367–401.

Kob72 S. Kobayashi, *T*ransformation Groups in Differential Geometry, (Springer-Verlag, New York, Heidelberg, Berlin, 1972).

Mac87 K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, (London Mathematical Society Lecture Note Series, 124. Cambridge University Press, Cambridge, 1987).

Ort06 E. Ortaçgil, preprint, math.DG/0604223

Sha97 R. W. Sharpe, *Differential geometry. Cartan's generalization of Klein's Erlangen program*, (Graduate Texts in Mathematics, 166. Springer-Verlag, New York, 1997).

Tel72 N. Teleman, *A characteristic ring of a Lie algebra extension. I, II*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **52** (1972), 498–506; 708–711.

Ter78 C. L. Terng, *Natural vector bundles and natural differential operators*, Amer. J. Math. **100** (1978), 775–828.

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