INFINITE SERIES

$\{a_n\}$ a sequence in $\mathbb{R}$.

$\sum_{n=1}^{\infty} a_n$: an ∞ series.

<u>defn:</u> $s_k = \sum_{n=1}^{k} a_n$, k-th partial sum.

If $(s_k)_{k=1}^{\infty}$ is convergent (divergent) then we say $\sum_{n=1}^{\infty} a_n$ is convergent (divergent).

If $s_k \longrightarrow +\infty$ then we write $\sum_{n=1}^{\infty} a_n = +\infty$.

<u>ex:</u> 1 If $\sum a_n$ is convergent then $a_n \longrightarrow 0$:

In $\mathbb{R}$ convergence of a sequence $\Longleftrightarrow$ Cauchy sequence

completeness

② $\sum_{n=1}^{\infty} ar^n$; a geometric series

$a, r \in \mathbb{R}$.

$$s_k = \frac{1-r^{k+1}}{1-r} \cdot a \xrightarrow[k\to\infty]{} l \in \mathbb{R}$$

require

$\Longleftrightarrow |r|<1$

③ Taylor's theorem provides a natural source for ∞ series which are convergent

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)$$

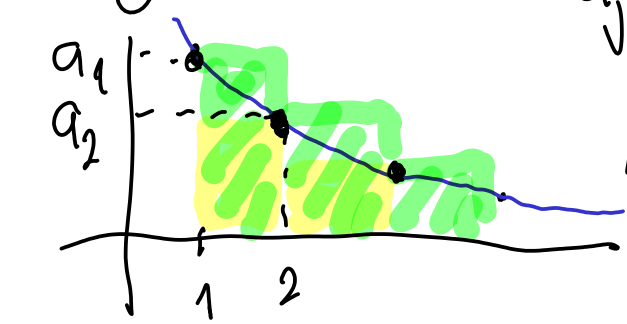
under some assumptions

PLAN:

integral
comparison
limit comp
root, ratio
Raabe's

} tests for series with positive terms

Integral test.



$a_j > 0$. Suppose there is some $\boxed{\text{decreasing}}$ $f : [1,\infty) \to \mathbb{R}^+$ s.t. $f(k) = a_k$.

$\sum a_n$ converges iff $\int_1^\infty f(x)\,dx$ converges.

Proof. $s_k = \sum_{n=1}^{k} a_n = a_1 + \sum_{n=2}^{k} a_n \le a_1 + \int_1^k f$

$\Leftarrow$: If $\int_1^\infty f$ converges then s_k is bounded from above.
By Monotone Convergence (s_k) converges.

$\Rightarrow$: If $\int_1^\infty f$ diverges:

$$s_k = \sum_{n=1}^{k-1} a_n + a_k \ge \int_1^k f + f(k) \longrightarrow \infty$$

- an approximation:

Observe in case of convergence, for any $N \in \mathbb{Z}^+$:

$$\int_N^\infty f \leq \sum_{n=N}^\infty a_n \leq a_N + \int_N^\infty f .$$

$$\Rightarrow 0 \leq \Big(\text{sum} - (a_1 + \cdots + a_{N-1})\Big) - \int_N^\infty f \leq a_N$$

Thus if one approximates the sum with $a_1 + \cdots + a_{N-1} + \int_N^\infty f$, the error made is $\leq a_N$.

Corol. $\sum_{n=1}^\infty n^{-p}$ converges iff $\int_1^\infty x^{-p}$ converges

iff $p > 1$.

ex: $\sum_{n=1}^\infty \frac{1}{n}$: harmonic series is divergent.

Comparison test.

Assume $a_n \geq b_n > 0$ for $n \geq 1$ (or ultimately). Then if $\sum a_n$ converges so does $\sum b_n$.

proof. Let s_k & t_k be k-th partial sums of $\sum a_n$ & $\sum b_n$ resp. $s_k \geq t_k$, $\forall k$. Since t_k is an increasing sequence & it is bounded from above by $s = \lim s_k$. By Mon. Seq., (t_k) converges too.

Limit comparison test.

Let $L = \lim \frac{a_n}{b_n}$ exists & $0 < L < \infty$. Then $\sum a_n$ & $\sum b_n$ both converge or both diverge.

If $L = \begin{matrix} 0 \\ \infty \end{matrix}$ and $\sum b_n$ $\begin{matrix} \text{converges} \\ \text{diverges} \end{matrix}$ so does $\sum a_n$.

proof.

$\frac{L}{2} < \frac{a_n}{b_n} < 2L$ ultimately. Hence $a_n < 2L b_n$ & $b_n < \frac{2}{L} a_n$.

If $L = 0$ then $a_n < \varepsilon b_n$. Similarly for $L = \infty$.

Ratio test.

Suppose $\frac{a_{n+1}}{a_n} \longrightarrow L$.

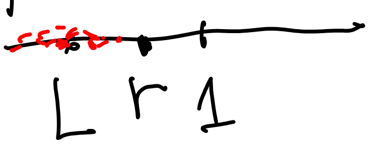
If $L<1$ then $\sum a_n$ converges.
$L>1$ diverges

For $L=1$ consider $\sum n^{-2}$; convergent & $\frac{(n+1)^{-2}}{n^{-2}} \to 1$

no conclusion possible

$\sum n^{-1}$; divergent. $\frac{(n+1)^{-1}}{n^{-1}} \to 1$.

proof.

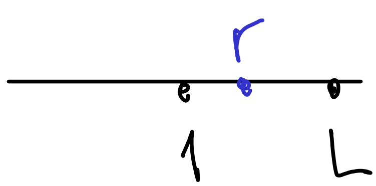


$\exists\ L<r<1$ for which

$\frac{a_{n+1}}{a_n} < r$. for $n \geq N$.

$a_{N+1} < r a_N$, $a_{N+2} < r a_{N+1} < r^2 a_N$, ...

$\sum_{n=N}^{\infty} a_n$ is convergent comparing with $\sum_{n=0}^{\infty} r^n a_N$.



$\exists\ 1<r<L$ s.t. $\frac{a_{n+1}}{a_n} > r$ ultim.

Then $a_{n+1} > r a_n > a_n$.

Since $a_n \not\to 0$, $\sum a_n$ is divergent.

Root test. Suppose $\sqrt[n]{a_n} \xrightarrow[n\to\infty]{} L$.

The same conclusion.

In case $L=1$ find two sequences, one is convergent, other divergent.

proof. $L<1$. $\sqrt[n]{a_n} < r < 1$, ultimately.

$$\Rightarrow a_n < r^n$$

By comparison with $\sum r^n$, $\sum a_n$ converges.

ex: $a_n = \dfrac{1\cdot 4\cdot 7\cdot \ldots \cdot (3n+1)}{n^2 \cdot 3^n \cdot n!}$

Ratio test: $\dfrac{a_{n+1}}{a_n} = \dfrac{\cancel{1\cdot 4}\cdots(3n+4)}{(n+1)^2\cdot 3^{\cancel{n+1}}\cdot \cancel{(n+1)!}} \cdot \dfrac{n^2 \cancel{3^n} \cancel{n!}}{\cancel{1\cdot 4\cdots(3n+1)}}$

$$= \frac{(3n+4)\cdot n^2}{3(n+1)\cdot(n+1)^2} \xrightarrow[n\to\infty]{} 1 = L \text{ no conclusion}$$

for the series $\sum a_n$.

Raabe's test. Suppose

$$\frac{a_{n+1}}{a_n} \longrightarrow 1 \quad \& \quad n\cdot\left(1-\frac{a_{n+1}}{a_n}\right) \xrightarrow[n\to\infty]{} L.$$

If $L>1$, $\sum a_n$ converges
$L<1$, $\sum a_n$ diverges.

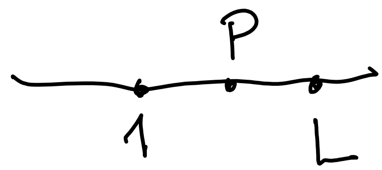
back to ex.

$$\left(1-\frac{a_{n+1}}{a_n}\right)\cdot n = n\cdot \frac{3(n+1)^3-(\cancel{3n^3}+4n^2)}{3(n+1)^3}$$

$$=\frac{(5n^2+\dots)\cdot n}{3(n+1)^3} \longrightarrow \frac{5}{3} > 1.$$

$\sum a_n$ is convergent.

proof of Raabe.



$\exists\ 1<P<L$ s.t. $n\left(1-\frac{a_{n+1}}{a_n}\right) > P$ ultimately.

$$\Rightarrow \frac{a_{n+1}}{a_n} < 1-\frac{P}{n} < \left(1+\frac{1}{n}\right)^{-P}$$

$$f(x)=(1+x)^{-P} \overset{\text{around } x=0}{=} 1-P\cdot x+ \text{ERR}\to 0$$

!

Recall $0 < E_{RR} = \frac{f''(c)}{2!}\cdot(x-0)^2 < p(p+1)\frac{x^2}{2}$

$(0 < c < x)$

$f''(x) = +p\cdot(p+1)\cdot(1+x)^{-p-2} < p(p+1)$

When $x = \frac{1}{n}$: $0 < E_{RR} < p(p+1)\frac{1}{2n^2} \xrightarrow[n\to\infty]{} 0$

with

$\frac{a_{n+1}}{a_n} < \left(1+\frac{1}{n}\right)^{-p} \iff \frac{a_{n+1}}{a_n} < \frac{(n+1)^{-p}}{n^{-p}}$

$\iff \frac{a_{n+1}}{(n+1)^{-p}} < \frac{a_n}{n^{-p}}$ i.e, $\frac{a_n}{n^{-p}}$ is decreasing

Hence converges. By limit comparison, since $\sum n^{-p}$ converges so does $\sum a_n$.
$(p>1)$

exercise: prove divergence when $L<1$.

§ Series with negative terms

defn: $\sum a_n$, $a_n \in \mathbb{R}$.

- If $\sum |a_n|$ is convergent, $\sum a_n$ is called <u>absolutely convergent</u>.
- If $\sum a_n$ is convergent but not absolutely then it is called <u>conditionally convergent</u>.

prop. If a series is abs conv then it is convergent.

proof. $\sum |a_n|$ is convergent

$\iff$ its partial sum is a convgnt sequence.

$\iff$ " is a Cauchy sequence:

Given $\varepsilon > 0$ $\exists N$ s.t $k, n > N$

$$|S_k - S_n| < \varepsilon$$

$$= |a_k| + |a_{k-1}| + \dots + |a_{n+1}|$$

tria. ineq. $\rightarrow \geq$

$$|a_k + a_{k-1} + \dots + a_{n+1}| = |s_k - s_n|$$

$\implies \{S_n\}$ is Cauchy $\iff \{S_n\}$ is convergent ∎

ex: $a_n = \frac{(-1)^{n+1}}{n}$

$$\sum_{n=1}^{k} a_n = +\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{k+1}}{k}$$

$$f(x) = \log(1+x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n + R_k(x_0) \quad |x_0| < |x|$$

Taylor expansion (around $x=0$)

This equality is guaranteed when $|x|<1$. When $x=-1$ the corresp series is divergent. The case $x=1$ is our question.

$$f^{(n)}(x) = \frac{(-1)^{n-1}\cdot(n-1)!}{(1+x)^n} \quad \& \quad |R_k(x_0)| \overset{\text{Lagrange remainder}}{=} \left|\frac{f^{(k+1)}(x_0)}{(k+1)!}\right| \ \exists |x_0|<|x|$$

$$= \left|\frac{(-1)^k}{k+1} \cdot \frac{1}{(1+x_0)^{k+1}}\right| \overset{\text{when } x=+1}{\leq} 1/(k+1) \xrightarrow[k\to\infty]{} 0$$

$$= \left(\sum_{n=1}^{k} \frac{(-1)^{n-1}}{n} x^n + R_k(x_0)\right) \xrightarrow{k\to\infty} \log(x+1), \quad R_k(x_0) \to 0$$

Hence at $x=+1$, the remainder $\to 0$ too so that

$$\sum_{n=1}^{\infty} a_n \to \log(1+x)\Big|_{x=1} = \log 2 .$$

defn. Let $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a permutation.
(1-1, onto)

$\sum a_{\sigma(n)}$ is called a <u>rearrangement</u> of $\sum a_n$.

Prop. If $\sum_{n=1}^{\infty} a_n$ is abs. convg, so is $\sum_{n=1}^{\infty} a_{\sigma(n)}$.
with S

$\to P_k = \sum_{n=1}^{k} |a_{\sigma(n)}|$

$\to s_k$: abs. values' partial sum

Proof. All the terms in P_k appear in the 1st k' terms of the initial series.

$$s_n \to S \quad , \quad s_n < S$$
increasing

$$P_k \leq s_{k'} < S \Rightarrow \forall k \; P_k < S.$$

Hence $P_k \xrightarrow[\text{seq.}]{\text{Monotone}} S' \leq S$.

Changing the roles of $\sum a_n$ & $\sum a_{\sigma(n)}$ we get $S \leq S'$. $\therefore S = S'$

thm: Let $\sum a_n$ be conditionally convergent.

ex: $\sum \frac{(-1)^n}{n}$

Given any $r \in \mathbb{R}$, there is some σ s.t.

$\sum_{n=1}^{\infty} a_{\sigma(n)}$ is convergent with sum $= r$.

For the proof let us first note:

lemma. (i) If $\sum a_n$ abs conv then $\sum a_n^+$ & $\sum a_n^-$ are convergent as well. Here

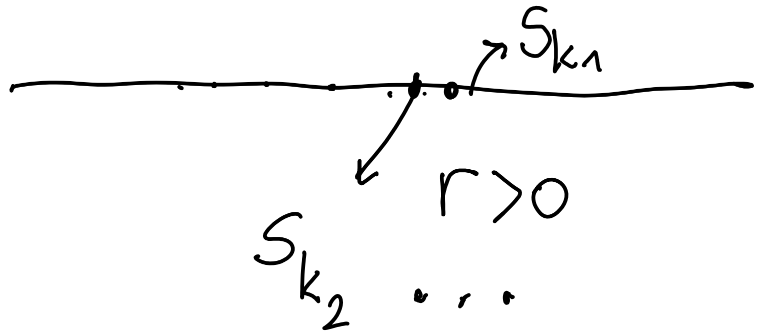
$$a_n^+ = \begin{cases} a_n & , a_n > 0 \\ 0 & , \text{otherwise} \end{cases} \qquad a_n^- = \begin{cases} |a_n| , & a_n < 0 \\ 0 , & \text{other} \end{cases}$$

(ii) If $\sum a_n$ is cond. conv then both $\sum a_n^+$ & $\sum a_n^-$ are divergent.

Proof: (i) $\forall n \; a_n^+, a_n^- \leq |a_n|$. By comparison we conclude.

(ii) $\sum |a_n|$ divergent $\Rightarrow$ One of $\sum a_n^+$ or $\sum a_n^-$ is divergent. (prove the contrapositive)

$\sum a_n$ convgnt $\Rightarrow$ Both are convgnt or divergent. (contrapositive) proof of thm. $\sum a_n^+$ & $\sum a_n^-$ are both divgnt



This rearrangement gives a series with sum converging to r: since $\sum a_n$ is convgnt

$a_n \longrightarrow 0$ ...

Dirichlet's Test

Consider $\{a_n\}$ & $\{b_n\}$.

- a_n is decreasing & $a_n \longrightarrow 0$.
- $\exists C \in \mathbb{R}^+$ s.t. the partial sums $|B_k| < C$.

Then $\sum a_n b_n$ is convergent.

ex: • $\sum \underbrace{\boxed{(-1)^n}}_{b_n} \cdot \underbrace{\frac{1}{n}}_{=a_n}$ is convergent.

• $\sum (-1)^n \cdot a_n$ is convergent. ← so-called alternating series test

Proof.
$$\sum_{n=0}^{k} a_n b_n = \underbrace{a_k(b_0+\dots+b_k)}_{a_k b_k}$$
$$+\ (\cancel{a_{k-1}}-a_k)\cdot\underbrace{(b_0+\dots+b_{k-1})}_{a_{k-1}b_{k-1}}$$
$$+\ (a_{k-2}-\cancel{a_{k-1}})\cdot(b_0+\dots+b_{k-2})$$
$$\vdots$$
$$= \underbrace{a_k\cdot B_k}_{\xrightarrow{k\to\infty}\ 0} + \sum_{n=0}^{k}(a_{n-1}-a_n)\cdot B_{n-1}$$

$$|a_k B_k| < C\cdot a_k \longrightarrow 0$$

$$\sum^{k} \overbrace{|(a_{n-1}-a_n)\cdot B_{n-1}|}^{\text{positive}}$$
$$= \sum^{k}(a_{n-1}-a_n)|B_{n-1}|$$
$$< C\sum_{0}^{k}(a_{n-1}-a_n)$$
$$< C(a_0-a_k) < Ca_0$$

upper bound

$\sum^{k}|(a_{n-1}-a_n)B_{n-1}|$ conveges as $k\to\infty$

$\Rightarrow \sum^{\infty}(a_{n-1}-a_n)B_{n-1}$ is abs conv.