Correction of the Solution of Exercise 5 of Chapter 2.5

Exercise 5 (Chapter 2.5 of R. Magnus' book). Recall the question: Let d and d' be metrics on a set X. Denote the open ball with centre a and radius r with respect to d by $B_r(a)$, and with respect to d' by $B_r^{'}(a)$. Prove that d and d' are equivalent metrics if and only if the following conditions are satisfied for every $a \in X$: for every $r > 0$, there is $s > 0$ so that $B_s^{'}(a) \subset B_r(a)$; and for every $r > 0$, there is $s > 0$ so that $B_s(a) \subset B_r^{'}(a)$.

Solution.($\Rightarrow$) In this direction, we assumed that the equivalence of d and d' implies that there exists $K_1, K_2 > 0$ such that

$$K_1 d'(x,y) \leq d(x,y) \leq K_2 d'(x,y) \quad \forall x, y \in X. \tag{1}$$

However, this is not true in general. The property (1) is a sufficient condition to say that d and d' are equivalent metrics, i.e., (1) implies that d and d' are equivalent; but the converse is not true.

Therefore, to prove the aforementioned direction, we cannot use (1). Now, let us assume that d and d' are equivalent. Then the identity map $I : (X,d) \to (X,d')$, $I(x) = x$ for all $x \in X$, is a homeomorphism. This reads that $I^{-1} : (X,d') \to (X,d)$ is also continuous. Using the continuity of I, at any $a \in X$ and given $r > 0$, there is $s_1 > 0$ such that $d(a,x) < s_1 \implies d'(I(a), I(x)) = d'(a,x) < r$. That is,

$$B_{s_1}(a) = I(B_{s_1}(a)) \subset B_r^{'}(a).$$

The first setwise identity is due to I being the identity function. The set inclusion above is the continuity of I. Conversely, as I^{-1} is also continuous, at any $a \in X$ and given $r > 0$, there is $s_2 > 0$ such that

$$B_{s_2}^{'}(a) = I^{-1}(B_{s_2}^{'}(a)) \subset B_r(a).$$

Hence, the proof of this direction is complete. There is no problem with the solution (discussed in PS) of the converse direction.