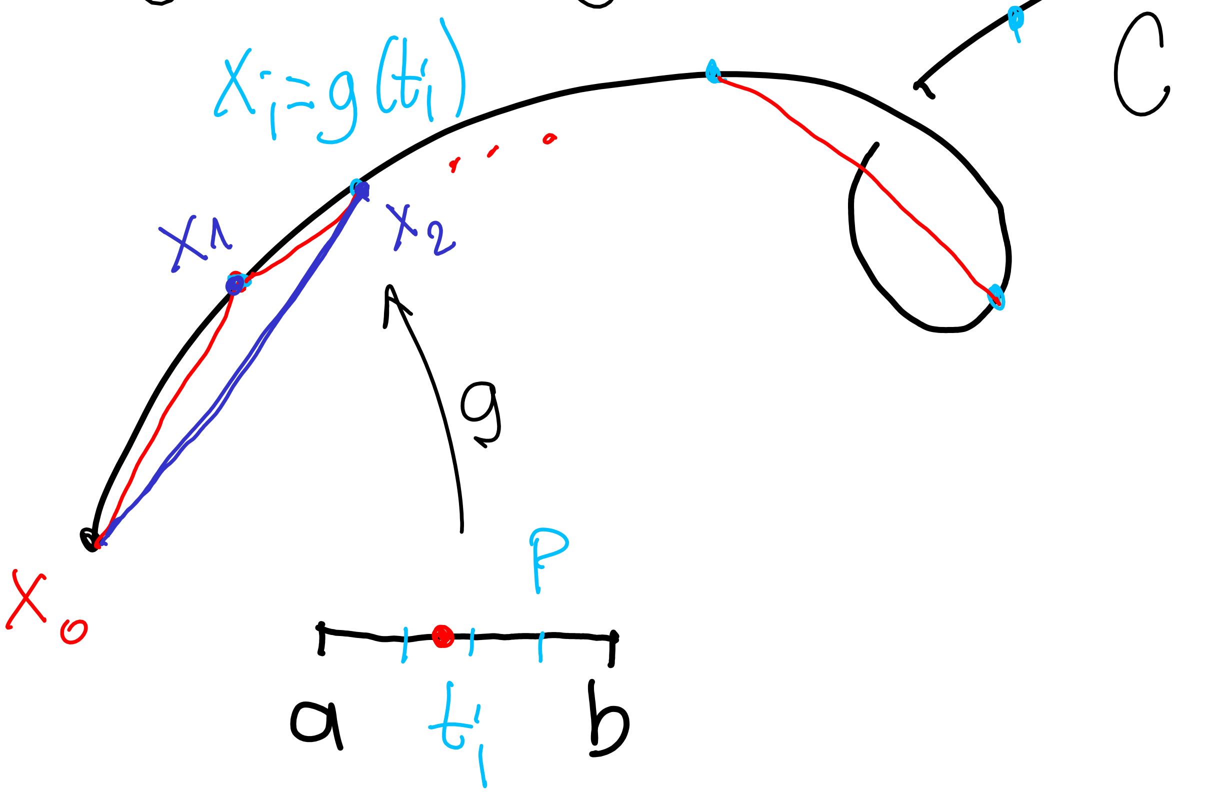
Integration along curves

LECTURE 10



$$L_P(C) \overset{\text{defn}}{=} \text{sum of lengths of line segments}$$

$$= \sum_{j=1}^{J} |x_j - x_{j-1}|$$

If P' is a partition of P then

$$L_{P'}(C) \geq L_P(C).$$

The set S of all such sums has a sup, which can be ∞. If S is bounded then C is called <u>rectifiable</u> (has length).

<u>Some facts</u>: Assume g is C^1.

① $\vec{g} : [a,b] \to \mathbb{R}^n$, $\vec{g}(t) = (x_1(t), \ldots, x_n(t))$

② $\displaystyle\int_a^b x_j'(t)\,dt \overset{\text{FTC}}{=} x_j(b) - x_j(a)$

③ $$\int_a^b g(t)\,dt = \int_a^b (x_1(t), \ldots, x_n(t))\,dt$$

$$\overset{\text{defn}}{=} \left(\int_a^b x_1(t)\,dt, \ \ldots, \ \int_a^b x_n(t)\,dt \right)$$

the integral of a vector-valued function.

④ $$\int_a^b g'(t)\,dt \overset{\text{defn}}{=} \left(\int_a^b x_1'\,dt, \ldots, \int_a^b x_n'\,dt \right)$$

$$\overset{\text{FTC}}{=} \left(x_1(b) - x_1(a), \ \ldots, \ x_n(b) - x_n(a) \right)$$

$$= \vec{g}(b) - \vec{g}(a)$$

thm: If g is C^1 then $L(C) = \int_a^b |g'(t)|\,dt$

$\hookrightarrow$ C is rectifiable and

proof. $L_p(C) = \sum_{j=1}^{J} |\vec{x}_j - \vec{x}_{j-1}|$, $\quad \vec{x}_j = \vec{g}(t_j)$

$$= \sum_{j=1}^{J} \left| \int_{t_{j-1}}^{t_j} g'(t)\,dt \right|$$

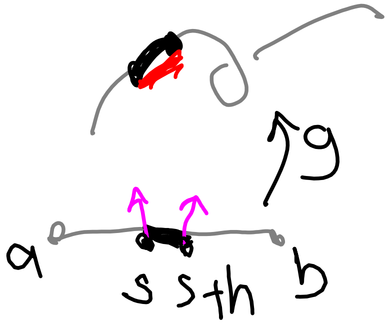
(2)

$$\leq \sum_{j=1}^{J} \int_{t_{j-1}}^{t_j} |g'(t)|\,dt = \underbrace{\int_a^b |g'(t)|\,dt}_{\text{defined \& finite}} = M \quad (*)$$

$\forall P$, $L_p(C) \leq M$ (bounded from above)

Hence C is rectifiable.

Now define $\varphi : \underset{[a,b]}{\underset{\Uparrow}{s}} \longrightarrow$ length from a to s



$$|g(s+h)-g(s)| \le |\varphi(s+h)-\varphi(s)| \overset{(*)}{\le} \int_s^{s+h} |g'(t)|\,dt$$

$$= |g'(\tau)| \cdot h$$

MVT: $\exists \tau \in (s, s+h)$

$$\left|\frac{g(s+h)-g(s)}{h}\right| \le \left|\frac{\varphi(s+h)-\varphi(s)}{h}\right| \le |g'(\tau)|$$

by continuity of g'

$h \to 0$

$\tau \to s$

As $h \to 0$:

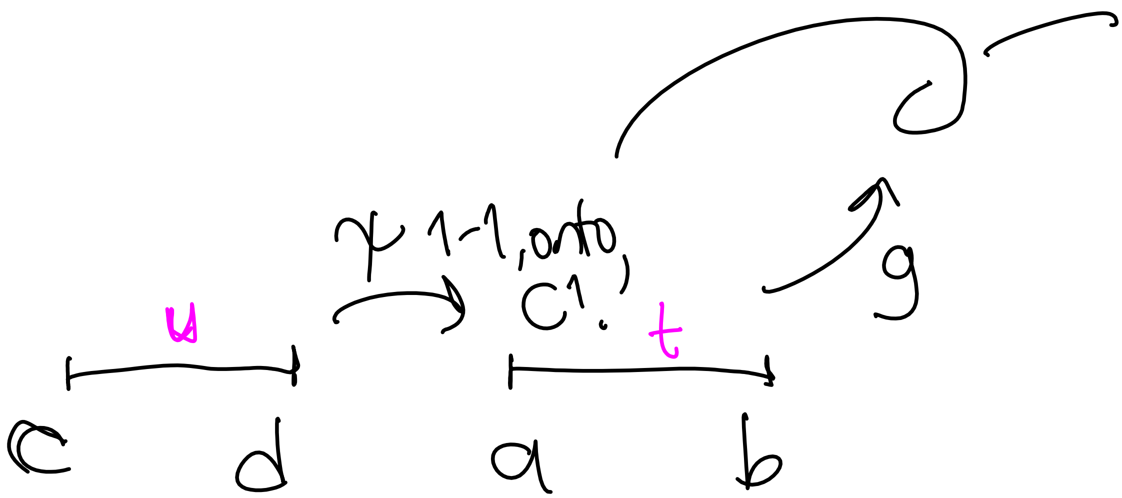
$$|g'(s)| = |\varphi'(s)| = |g'(s)|$$

$$\varphi(b) - \overset{=0}{\varphi(a)} = \text{length of } C.$$

$$= \int_a^b |\varphi'(t)|\,dt = \int_a^b |g'(t)|\,dt$$

Observation.

① Length does not depend on the param'n.



The new param'n: $g \circ \psi$.

$$L^g(C) = \int_a^b |g'(t)|\,dt = \int_c^d |g'(\psi(u))| \cdot |\psi'(u)|\,du$$

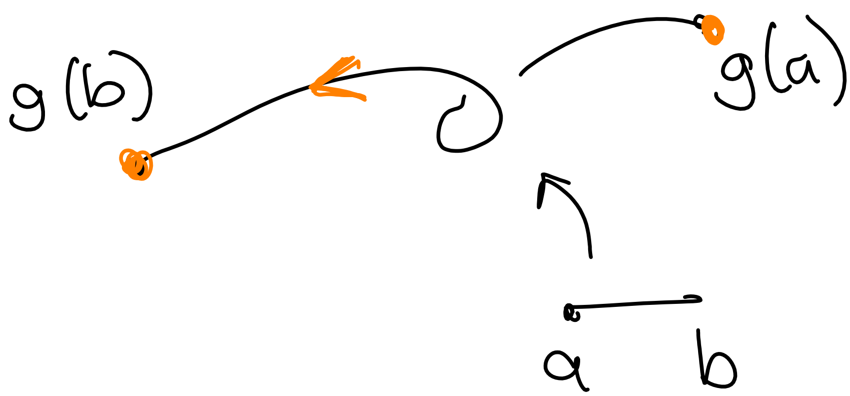
Change of variables

Chain Rule

$$= \int_c^d |(g \circ \psi)'(u)|\,du = L^{g\circ\psi}(C)$$

② Orientation:

A param'n determines an orientation of the curve, in the sense that it orders the end pts of the curve:



③ One can define integrals over curves:

Around C, $f : \underset{\substack{\downarrow \\ C}}{U} \to \mathbb{R}$

$$\int_C f\,ds \underset{\substack{\uparrow \\ g\ \text{param'n}}}{\overset{\text{defn}}{=}} \int_a^b f(g(t)) \cdot |g'(t)|\,dt$$

④ $L(C) = \int_a^b |g'(t)|\,dt = \int_a^b ds$

5 The line integral of f: $\int_C f\,ds$ is independent from the param'n.

Proof as in 1.