Boğaziçi University
Department of Mathematics
Math 231 Advanced Calculus I
Fall 2025 – Final Exam

1	2	3	4	$\sum$
25 pts	25 pts	25 pts	25 pts	100 pts

Date: December 26, 2025 Time: 17:15-19:15	Full Name: PROPOSED SOLUTIONS

1. Find the absolute extreme values, if any, of the function $k(x,y) = 3x^2 - 8xy - 4y^2 + 2x + 16y$ on the set $S = \{(x,y) : xy \geq 1\}$. If absolute min or max does not exist, give the reason explicitly and in detail.

Extreme values of k occur either at critical points or on ∂S, if they occur at all.

Observe that fixing y, say $y=1$, we get

$$\lim_{\substack{x \to +\infty \\ y=1}} k(x,y) = \lim_{x \to +\infty} 3x^2 - 6x + 12 = +\infty$$

Similarly let $x=1$:

$$\lim_{\substack{y \to +\infty \\ x=1}} k(x,y) = -\infty$$

Hence k does not have absolute extrema over S.

2. (a) [10] State the following Implicit Function Theorem, by completing the text that I started below:

Consider the function $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and a point $(\boldsymbol{a}, b) \in \mathbb{R}^n \times \mathbb{R}$. Suppose that

F is of class C^1, $F(\vec{a},b)=0$ and $\partial_{n+1} F(\vec{a},b) \neq 0$.

Then there are open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}$ with $\vec{a} \in U$, $b \in V$ and a C^1 function $f: U \to V$ such that $F(\vec{x}, f(\vec{x})) = 0$.

Moreover $\partial_j f(\vec{a}) = \partial_j F(\vec{a},b) / \partial_{n+1} F(\vec{a},b)$ $(1 \leq j \leq n)$

(b) (Buck, p366) [15] Let φ be a function of one variable for which $\varphi(1) = 0$. What additional conditions on φ will allow the equation $2\varphi(xy) = \varphi(x) + \varphi(y)$ to be solved for y in a neighborhood of $(1, 1)$? (Hint: In order to use part (a), start with defining a suitable function F.)

(b) Define $F: \mathbb{R}^2 \to \mathbb{R}$, $F(x,y) = 2\varphi(xy) - \varphi(x) - \varphi(y)$.
Observe $F(1,1)=0$. The question is to solve $F(x,y)=0$ for y in terms of x near the point $(1,1)$.

- Suppose φ is C^1. Then F is C^1 too.

We also need a derivative condition to call Imp Fnc Thm for help.
We need $0 \neq \partial_y F(1,1) = 2x\varphi'(xy) - \varphi'(y) \Big|_{(1,1)} = 2\varphi'(1) - \varphi'(1) = \varphi'(1)$

- So suppose $\varphi'(1) \neq 0$.

Then the result follows from Imp Fnc Thm.

3. (Folland, p120) Determine two of the variables x, y, z, t such that the general Implicit Function Theorem does not guarantee to solve the equations $z^3 + xt - y = 0, t^3 + yz - x = 0$ for those two variables as functions of the other two near the point $(x, y, z, t) = (0, -1, -1, -1) \in \mathbb{R}^4$.

Set $F(x,y,z,t) = (z^3 + xt - y \ , \ t^3 + yz - x)$, $F: \mathbb{R}^4 \to \mathbb{R}^2$.

Observe $F(P) = (0,0)$, and having polynomial components, F is C^1.

To be able to solve $F = (0,0)$ for x_1, x_2 as functions of the other two variables, the ImpFnc Thm requires

$$\begin{vmatrix} \partial_{x_1} F_1 & \partial_{x_2} F_1 \\ \partial_{x_1} F_2 & \partial_{x_2} F_2 \end{vmatrix} (P) \neq 0 .$$

This is not satisfied for $(x_1, x_2) = (x, y)$:

$$\det = \begin{vmatrix} t & -1 \\ -1 & z \end{vmatrix}_P = zt - 1 \Big|_P = 0$$

Thus the ImpFncThm does not guarantee that the equations can be solved for x, y as functions of z, t near P.

4. Here is the Inverse Function Theorem:
Consider a function $g : \mathbb{R}^k \to \mathbb{R}^k$ of class C^1. Let $\boldsymbol{u} \in \mathbb{R}^k$, $\boldsymbol{v} = g(\boldsymbol{u})$. Suppose $\det(Dg(\boldsymbol{u}))$ is nonzero. Then there are open sets $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^k$ containing $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively, and a function $h : V \to U$ of class C^1 such that h is the inverse function of $g : U \to V$. Moreover $Dh(\boldsymbol{v}) = (Dg(\boldsymbol{u}))^{-1}$.
This theorem can be proven directly using the general Implicit Function Theorem.

I am going to try to contradict with the Inverse Function Theorem. Consider the function

$$g : \mathbb{R} \to \mathbb{R}, g(x) = x + 2x^2 \sin \frac{1}{x} \text{ when } x \neq 0 \text{ and } g(0) = a.$$

(a) [4] What is the value of a which makes g a continuous function everywhere?
(b) [4] Show that the continuous function g in part (a) is differentiable at 0 and that $g'(0) \neq 0$.
(c) [7] Show that the function g in part (a) is not one-to-one in any neighborhood $(-\varepsilon, +\varepsilon)$ of 0.
(d) [10] Part (c) shows that whatever open neighborhood I of 0 you take, g cannot have an inverse over I. But by part (b), $g'(0) \neq 0$. How is this example not in contradiction with the Inverse Function Theorem? Prove your claim(s) explicitly.

(a) Observe $\lim_{x\to 0} g(x) = \lim_{x\to 0} x + 2x^2 \sin\frac{1}{x} = 0$, since $-1 \leq \sin 1/x \leq 1$.

So setting $g(0)=0$ make g continuous at 0.

(b) $\lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{1}{h}\left(h+2h^2 \sin\frac{1}{h}\right) = 1$. So $g'(0) = 1$.

(c) Waits for your creativity.

(d) No contradiction with the InvFncThm because g is not C^1 at 0; $g'(x) = \begin{cases} 1 & , x=0 \\ 1+4x\sin 1/x - 2\cos 1/x & , x \neq 0 \end{cases}$ is not cont. at 0:

$\lim_{x\to 0}\left(1+4x\sin\frac{1}{x} - 2\cos\frac{1}{x}\right) = 1 - 2\lim_{x\to 0}\cos\frac{1}{x}$ does not exist.