**Boğaziçi University**
**Department of Mathematics**
Math 231 Advanced Calculus I
**Fall 2025 – First Midterm Exam**

| 1 | 2 | 3 | 4 | $\sum$ |
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| 28 pts | 25 pts | 25 pts | 22 pts | 100 pts |

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| Date: October 31, 2025<br>Time: 17:00-19:00 | Full Name: PROPOSED SOLUTIONS |

1. Prove or give a counter-example:

(a) Every bounded sequence in $\mathbb{R}^k$ is Cauchy.

FALSE: $a_n = (-1)^n$ bounded. For $\forall i$ odd, $j$ even $|a_i - a_j| = 2$.
So given $\varepsilon < 2$, one cannot find $N$ to satisfy the defn of being Cauchy.

(b) Consider a sequence $(u_n)_{n=1}^{\infty}$ in $\mathbb{R}^k$. If $||u_{n+1} - u_n|| \longrightarrow 0$ as $n \longrightarrow \infty$ then the sequence $(u_n)$ is convergent.

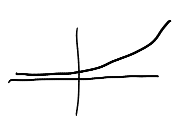
FALSE: Consider $u_n = \sum_{k=1}^{n} \frac{1}{k}$. Then $u_{n+1} - u_n = \frac{1}{n+1} \xrightarrow{n\to\infty} 0$
but $\forall M > 0$ $\exists$ some $N$ : $u_N = \sum_{k=1}^{N} \frac{1}{k} > M$.
Hence $(u_n)$ diverges (to $+\infty$).

(c) If a nonempty set $A \subset \mathbb{R}^k$ is bounded then $\operatorname{diam}(A) \in \mathbb{R}$. (Recall: $\operatorname{diam}(A) = \sup\{||x-y|| : x, y \in A\}$.)

TRUE: $A$ bdd $\Leftrightarrow A \subset B(R,0)$. Then $\forall x,y \in A$, $x,y \in B(R,0)$ so that
$\|x-y\| \leqslant \|x\| + \|y\| \leqslant 2R$. Hence $\operatorname{diam} A \leqslant 2R \in \mathbb{R}$.

(d) Let $n, k \geq 1$ and $f : \mathbb{R}^n \to \mathbb{R}^k$ be defined **over all** $\mathbb{R}^n$. If $f$ is **continuous** and $D \subset \mathbb{R}^n$ is **closed**, then $f(D)$ is closed in $\mathbb{R}^k$.

FALSE: $n = k = 1$ ; $f(x) = e^x$.



$D = \mathbb{R}$ closed in $\mathbb{R}$;
$f(\mathbb{R}) = (0, +\infty)$ not closed in $\mathbb{R}$.

2. For a bounded sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$, define the set $S = \{x \in \mathbb{R} : x < a_n \text{ for infinitely many } n\text{'s}\}$.

(a) Show that $S$ is nonempty by explicitly giving an element in $S$.

$(a_n)$ is bounded. Any lower bound of the set $\{a_n\}$ is in $S$.

(b) Why is $\sup S$ finite? (Denote this sup by $a$.)

$S$ is bounded from above by any upper bound of $\{a_n\}$.
Then by Completeness, sup exists.

(c) Show that $(a_n)$ has a subsequence which converges to $a$. Do this by first constructing a subsequence $(a_{n_k})_{k=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ (tell why the indices can be chosen in an increasing manner); then you must show that the subsequence you constructed converges to $a$. (Prove all these from scratch! Do not use Bolzano-Weierstrass theorems here. Because your proof will be a new proof of the Bolzano-Weierstrass Theorem I: *Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence.*)

$a = \sup S$. Equivalently $a$ is an upper bound for $S$ and $\forall \varepsilon > 0$ $(a-\varepsilon, a]$ contains an element of $S$. So $\forall k$, choose an element $s_k$ of $S$ in $(a-\frac{1}{k}, a]$.

If there is not $\infty$ly many $a_n$'s $< a$ then there's a decreasing subseq of $(a_n)$ with $\lim = a$.

Otherwise $s_k \in S \iff \exists\, \infty$ly many elts of $(a_n)$ in $(a-1/k, a]$. Choose one with index $n_k$ greater than the previously chosen $n_{k-1}$.

<u>claim.</u> The constructed $(a_{n_k})_{k=1}^{\infty}$ converges, with limit $= a$. PROVE IT!

3. Consider a sequence $(b_n)_{n=1}^{\infty}$ in $\mathbb{R}^k$, $k \geq 1$. Suppose there is some real $c \in (0,1)$ such that for all $n$, $|b_{n+1} - b_n| \leq c|b_n - b_{n-1}|$. Prove that the sequence $(b_n)$ is convergent. (Hint: • Bound the differences in terms of $|b_2 - b_1|$. • Recall the sum formula for $1 + r + \ldots + r^k$. • Cauchy. )

Observe $\forall j \geq 2$: $|b_{j+1} - b_j| \leq c \cdot |b_j - b_{j-1}| \leq \cdots \leq c^{j-1} \cdot |b_2 - b_1|$.

Then $\forall i > j \geq 2$: $|b_i - b_j| \leq \underbrace{|b_i - b_{i-1}| + \cdots + |b_{j+1} - b_j|}_{\text{triangle ineq}}$

$\leq (c^{i-2} + c^{i-3} + \cdots + c^{j-2}) \cdot |b_2 - b_1|$

$= \frac{1 - c^{i-1} - (1 - c^{j-2})}{1-c} \cdot \underbrace{\frac{|b_2 - b_1|}{1-c}}_{= \Delta : \text{fixed}}$

$= c^{j-2} \cdot (1 - c^{i-j+1}) \cdot \Delta \leq c^{j-2} \cdot \Delta$

Thus given $\varepsilon > 0$, choose $N$ s.t. $c^{N-2} \cdot \Delta < \varepsilon$

i.e. $N - 2 > \frac{\log \varepsilon/\Delta}{\log c}$

so that $\forall i,j > N$: $|b_i - b_j| \leq c^{j-2} \cdot \Delta < c^{N-2} \Delta < \varepsilon$

So $(b_n)$ is Cauchy, therefore convergent in $\mathbb{R}$.

4. Let $K$ be closed and $P$ be open in $\mathbb{R}^n$. Then $K - P$ is <u>closed</u> in $\mathbb{R}^n$. Fill in the blank and prove that statement.

Observe $K - P = K \cap P^c$ is closed because $K$ & $P^c$ are closed.