Ch. 7.1:

Ex. 3: Let A and B be connected sets in a m.s. X, and suppose that $A \cap \overline{B} \neq \phi$. Show that $A \cup B$ is connected.

Solution: Suppose for a contradiction that $A \cup B$ is disconnected. Then there are $\mathcal{U}, V$ open, nonempty and disjoint subsets of X such that we have

1) $(A \cup B) \cap \mathcal{U} \neq \phi$, 2) $(A \cup B) \cap V \neq \phi$ and 3) $A \cup B \subseteq \mathcal{U} \cup V$.

As A and B are connected, they are completely contained in one of $\mathcal{U}$ and V. Assume wlog that $A \subset \mathcal{U}$ and $B \subset V$. (if $B \subset \mathcal{U}$, then $A \cup B \subset \mathcal{U}$ and this would imply that $V = \phi$ which is a contradiction)

As $A \cap \overline{B} \neq \phi$, let $x \in A \cap \overline{B}$, i.e., $x \in A$ and $x \in \overline{B}$. Then $x \in \mathcal{U}$ since $A \subset \mathcal{U}$.

If $x \in \overline{B}$, then there is a sequence $(x_n)_{n=1}^{\infty} \subset B$ such that $x_n \to x$ as $n \to \infty$. Thus, $x_n \in V$ for all n and by the same reasoning $x \in \overline{V}$. Thus, $x \in \mathcal{U} \cap \overline{V}$.

Remember that $\mathcal{U} \cap V = \phi$ and if $x \in \mathcal{U}$, then there is $r > 0$ such that $B_r(x) \subset \mathcal{U}$
$\Rightarrow B_r(x) \cap V = \phi \Rightarrow x \notin \overline{V} \Rightarrow \mathcal{U} \cap \overline{V} = \phi$. Thus, there is no $x \in \mathcal{U} \cap \overline{V}$ which says that $A \cap \overline{B} = \phi$ which is a contradiction. Therefore, $A \cup B$ is connected.

Ex. 4: Let $(M_i)_{i \in I}$ be a family of connected sets in X s.t. for every partition of index set $I = I_1 \cup I_2$ $(I_1 \cap I_2 = \phi)$, we have

$$\bigcup_{i \in I_1} M_i \cap \bigcup_{j \in I_2} M_j \neq \phi \ldots (*)$$

Then $\bigcup_{i \in I} M_i$ is connected.

Solution: Suppose for a contradiction that $\bigcup_{i \in I} M_i =: M$ is disconnected. Then there are open disjoint subsets $\mathcal{U}, V$ of X such that

1) $M \subset U \cup V$, 2) $M \cap U \neq \phi$, and 3) $M \cap V \neq \phi$.

Also, given $i \in I$, M_i is completely contained in either U or V, since ... Define $I_1 \subset I$ and $I_2 \subset I$ as otherwise M_i would be disconnected.

$I_1 := \{ i \in I : M_i \subset U \}$ and $I_2 := \{ i \in I : M_i \subset V \}$.

Then it is clear by the above observation that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = I$.

As $\bigcup_{i \in I_1} M_i \subset U$, $\bigcup_{j \in I_2} M_j \subset V$ and $U \cap V = \emptyset$, we get

$$\bigcup_{i \in I_1} M_i \cap \bigcup_{j \in I_2} M_j = \emptyset \text{ which contradicts } (*).$$

Ex. 6: Show that a m.s. X is connected iff every continuous function from X to $\{0,1\}$ with the discrete metric is constant.

Solution: Solving this is left as an exercise for you. Here are the hints:

1) $(\Rightarrow)$ Prove the contrapositive statement. Suppose there is a nonconstant continuous function $f : X \to \{0,1\}$. The task is to show that X is disconnected.

2) Define the sets $A := f^{-1}(\{0\})$ and $B := f^{-1}(\{1\})$. Argue that they are open, disjoint subsets of X such that $X = A \cup B$.

3) $(\Leftarrow)$ Again prove the contrapositive statement. Suppose that X is disconnected. Then there are disjoint open subsets A, B of X such that $X = A \cup B$. Let us define $f : X \to \{0,1\}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B. \end{cases}$$

Show that f is a nonconstant continuous map from X to $\{0,1\}$. You prove continuity of f by showing that preimage of every open subset of $\{0,1\}$ is open in X. Start identifying the complete collection of open subsets of $\{0,1\}$ with respect to discrete metric.

Ch. 7.2:

... that a normed vector space is connected.

Ex. 1 : Show that a norme

Solution : Solving this is an exercise for you. Here are the guidelines:

1) Let X be a normed vector space and $x, y \in X$. As X is a vector space, we have $tx + (1-t)y \in X$ for all $0 \le t \le 1$.

2) Define the function $f : [0,1] \to X$ by $f(t) = tx + (1-t)y$. Show that f is a continuous curve in X. Then conclude that X is arcwise connected, hence connected.

Ex. 4 : Let A be a set in a m.s. X.

(a) Show that if A is closed then $(\partial A)^\circ = \phi$.

Solution : Showing this is an exercise for you. Here are the hints:

1) Suppose for a contradiction that $(\partial A)^\circ \neq \phi$. Then pick $x \in (\partial A)^\circ$ and observe that there is $r > 0$ such that $B_r(x) \subset \partial A$.

2) As A is closed, we have $\bar{A} = A \Rightarrow \partial A \subseteq A$. Then $B_r(x) \subset A$. Since $\partial(A^c) = \partial A$, conclude also that $B_r(x) \subset \overline{A^c}$. Argue further that

$$B_r(x) \cap A^c \neq \phi.$$

3) By the observations in part (2), you have that $B_r(x) \subset A$ and $B_r(x) \cap A^c \neq \phi$. This is a contradiction. Thus, conclude that there is no such $x \in (\partial A)^\circ$, i.e., $(\partial A)^\circ = \phi$.

(b) Show that $\partial\partial\partial A = \partial\partial A$.

Solution : Recall that $\partial A = \bar{A} \setminus A^\circ \Rightarrow \partial A$ is closed. Thus, boundary of any set is closed. Then since $\partial\partial A$ is closed, it is clear that $\partial\partial\partial A \subseteq \partial\partial A$. Moreover, by definition

$$\partial\partial\partial A = \overline{\partial\partial A} \setminus (\partial\partial A)^\circ = \overline{\partial\partial A} = \partial\partial A \quad \text{as } \partial\partial A \text{ is closed and } (\partial\partial A)^\circ = \phi$$

by part (a).

Ch. 7.3

Ex. 5: Suppose that X is a m.s. and the number of connected components is finite. Show that each connected component is open.

Solution: Let us denote $X_1, \dots, X_n$ as connected components of X, i.e., $X = \bigcup^{n} X_i$ and $X_i \cap X_j = \emptyset$ for $i \neq j$. Let $x \in X_1$, and set

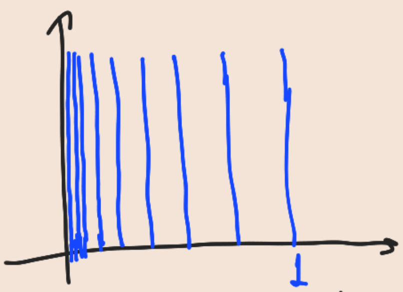
$$r_j = d(x, X_j) = \inf\{ d(x,y) : y \in X_j \} \text{ for } 2 \leq j \leq n.$$

Claim: $r_j > 0$ for all $2 \leq j \leq n$.

Proof: Suppose that $r_{j_0} = 0$ for some j_0. Then $d(x, X_{j_0}) = 0$ and since X_{j_0} is closed, we necessarily have $x \in X_{j_0}$, which is a contradiction.

Next, define $r := \frac{1}{3} \min\{r_1, \dots, r_n\}$. Then $B(x,r) \subset X_1$ since otherwise $B(x,r) \cap X_{j_0} \neq \emptyset$ for some j_0, which implies that there is $z \in X_{j_0}$ such that $d(x,z) < r \leq \frac{1}{3} r_{j_0}$ which contradicts the fact that $d(x, X_{j_0}) = r_{j_0}$. Thus, x is an interior point of X_1. As x and X_1 are chosen arbitrarily, we are done.

Ex. 10: Recall topologist's comb set $X = \bigcup_{n=1}^{\infty} (\{1/n\} \times [0,1])$ that is viewed as a subspace of $\mathbb{R}^2$.



- Sketch of the set

a) Show that each segment $\{1/n\} \times [0,1]$ is a connected component of X.

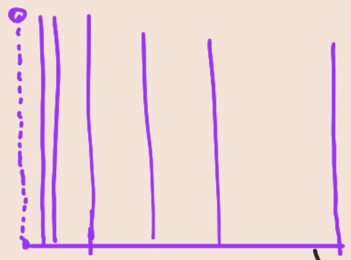
Solution: It is clear that $\{\frac{1}{m}\} \times [0,1]$ and $\{\frac{1}{n}\} \times [0,1]$ are disjoint whenever $m \neq n$. Also, $\{1/n\} \times [0,1]$ is pathwise connected for every $n \in \mathbb{N}_+$, hence connected.

It suffices to show the maximality of the sets. Pick $n \in \mathbb{N}_+$ and fix it. Suppose for some $x \in \{1/n\} \times [0,1]$ that the connected component C of X ... $\in \mathbb{N}_+$, $m \neq n$ such that is bigger than $\{1/n\}\times[0,1]$. Then there is $m\in\mathbb{N}^+$, $m\neq n$, such that $(\{1/m\}\times[0,1])\cap C\neq\emptyset$. But then $(\{1/n\}\times[0,1])$ and $(\{1/m\}\times[0,1])\cap C$ are disconnections of C, contradiction.

The last argument might be wrong. Instead use the definition of connected component of $x\in\{1/n\}\times[0,1]$, which is the set of all points contained in a connected subset of X together with x:
$C_x=\{y\in X : x \text{ and } y \text{ are contained in some connected set in } X\}$

(b) Append the segment $[0,1]\times\{0\}$ to X and call the new space X_1. Show that X_1 is connected, in fact, arcwise connected.

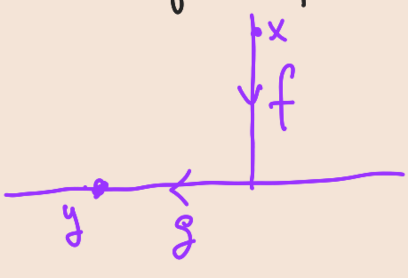
Solution:



Let $x,y\in X_1 = X\cup([0,1]\times\{0\})$. If both $x,y\in[0,1]\times\{0\}$ then we can join them by the function $f(t)=tx+(1-t)y$ for $0\le t\le 1$. Clearly f is continuous. If $x\in X$ and $y\in[0,1]\times\{0\}$, then there is $n\in\mathbb{N}^+$ such that $x\in\{1/n\}\times[0,1]$. Define

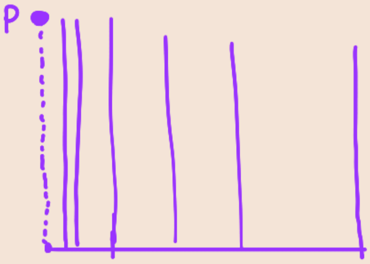
$f:[0,1]\to\{\frac{1}{n}\}\times[0,1]$ as $f(t)=(\frac{1}{n},(1-t)x_2)$ (vertical down)

Also define $g:[0,1]\to[0,1]\times\{0\}$ as $g(t)=t(\frac{1}{n},0)+(1-t)y$ (horizontal).
g is continuous as well. If we glue f and g, we again obtain a continuous path from x to y:



The case where x and y are left as exercise for you.

c) Append the point $p:=(0,1)$ to X_1, call the new space X_2. Show that X_2 is connected.



— X_2 space —

Solution: Consider the closure of X_2.

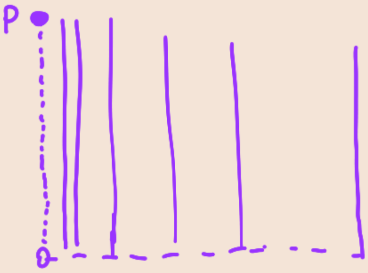
$$\overline{X_2} = \overline{X_1} = \bigcup_{n=1}^{\infty} \{\tfrac{1}{n}\}\times[0,1] \cup [0,1]\times\{0\} \cup \{0\}\times[0,1].$$

It can be shown that $\overline{X_1}=\overline{X_2}$ is arcwise connected by the same argument in part (b). We then have

$$X_1 \subset X_2 \subset \overline{X_1}$$ with $X_1, \overline{X_1}$ connected sets. Then X_2 is connected as well.

d) Remove the segment $[0,1)\times\{0\}$ from X_2, call the new space X_3. Show that $\{p\}$ is a connected component of X_3.

Solution:



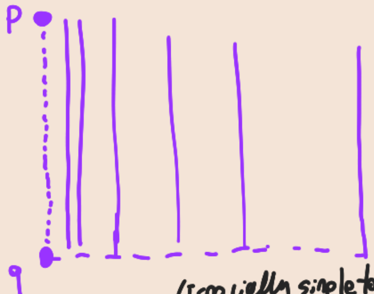
— X_3 space —

$$X_3 = X \cup \{p\} = \{p\} \cup \bigcup_{n=1}^{\infty} \{1/n\}\times[0,1].$$

Clearly X_3 is disconnected and there is no $n\in\mathbb{N}_+$ such that $p \in \{1/n\}\times[0,1]$. Thus, $\{p\}$ is the maximal connected subset of X_3 containing p.

e) Append the points $p=(0,1)$ and $q=(0,0)$ to X, call the new space X_4. Show that $\{p\}$ and $\{q\}$ are connected components of X_4. Show however that it is not possible to express X_4 as a union $A\cup B$, where $A,B\subset X_4$ open in X_4, $A\cap B=\emptyset$, $p\in A$ and $q\in B$.

Solution:

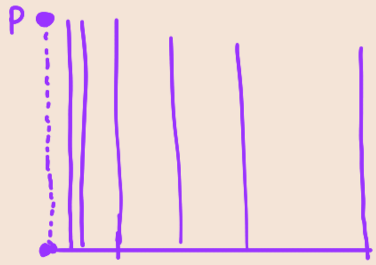


Showing that $\{p\}$ and $\{q\}$ are connected components of X_4 is similar to part (d). Notice that for any $r>0$, both $B_r(p)$ and $B_r(q)$ intersect with infinitely many line segments $\{1/n\}\times[0,1]$ of X_4. Moreover, all connected components (especially singletons) $\{p\}, \{q\}, (\{1/n\}\times[0,1])_{n=1}^{\infty}$ are closed but not all of them are open in X_4. Thus, we cannot separate p and q by disjoint open subsets A, B of X_4 such that $A\cup B = X_4$.

$A \cup B = X_4$.

f) Show that the connected space X_2 defined above is not arcwise connected. Identify its arcwise connected components.

Solution: Let us consider the pair of points $(0,1)=p$, $q=(0,0) \in X_2$. There is no continuous path joining p to q as for every $m,n \in \mathbb{N}_+$, there is $r \in \mathbb{R}\setminus\{0\}$ such that $\frac{1}{m} < \frac{1}{r} < \frac{1}{n}$. Thus, $\{\frac{1}{r}\} \times [0,1] \subset X_2^c$. As in part (b), we cannot continuously move vertically or horizontally.



- X_2 space -

Ch. 1.3:

Ex. 1: Show that Cantor's middle thirds set $B \subset [0,1]$ does not include any open interval (c,d), $c<d$.

Solution: Recall that B consists of points in $[0,1]$ having the ternary representation $\sum_{k=1}^{\infty} \frac{a_k}{3^k}$ where $a_k \in \{0,2\}$. By construction, B is a countable intersection of closed sets, i.e., it is a closed set. To prove the claim it suffices to prove that B has no interior points, i.e., $\mathring{B} = \emptyset$.

For a contradiction, we assume that $\mathring{B} \neq \emptyset$ and let $x \in \mathring{B}$. Recall that given $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that $3^{-n} < \varepsilon$. So, by changing the n-th digit of ternary representation of x from 0 or 2 to 1, say this number $\tilde{x}$, we obtain a number which does not belong to B. However, $\tilde{x} \in (x-\varepsilon, x+\varepsilon)$. This shows that there is no $\varepsilon > 0$ so that $x \in B$ implies $(x-\varepsilon, x+\varepsilon) \subset B$. Thus, $\mathring{B} = \emptyset$.

Ch. 3.2:

Ex. 2: The space $2^{\mathbb{N}_+} := \{ (x_n)_1^\infty : x_n \in \{0,1\} \text{ for all } n \in \mathbb{N}_+ \}$ is homeomorphic to an ultrametric space. For two sequences $a, b \in 2^{\mathbb{N}_+}$, we set $d(a,b) = 2^{-n}$, where n is the lowest place number at which a and b differ.

(a) Show that d is an ultrametric.

Solution: It is clear that $d: 2^{\mathbb{N}_+} \times 2^{\mathbb{N}_+} \to [0, \infty)$.

1) If $a=b$, then $d(a,b)$. Suppose that $d(a,b)=0$ for some $a,b \in 2^{\mathbb{N}_+}$. Let n be the lowest number place at which a and b differ. Then

$d(a,b) = 2^{-n} > 0$, which is a contradiction, i.e., there is no $n \in \mathbb{N}_+$ so that n-th digit of a and b differ. Thus, $a=b$.

2) Clearly, $d(a,b) = d(b,a)$

3) Let $a,b,c \in 2^{\mathbb{N}_+}$ so that $d(a,c) = 2^{-n}$ and $d(b,c) = 2^{-m}$ for some $m,n \in \mathbb{N}_+$. Note that if a and c start differing at n-th digit and the same holds at the m-th digit for b and c, we infer that $\min(m,n)$ is the lowest digit at which a and b differ. So,

$$d(a,b) = 2^{-\min(m,n)} \leq \max(2^{-m}, 2^{-n}) = \max(d(a,c), d(b,c)).$$

Thus, d is an ultrametric.

(b) Show that d is equivalent to any of the usual product metrics.

Solution: Suppose that the sequence $\{a^{(i)}\} \subset 2^{\mathbb{N}_+}$ converges to $b \in 2^{\mathbb{N}_+}$ with respect to the ultrametric d. That is, there is some $N \in \mathbb{N}_+$ such that

$$d(a^{(i)}, b) < 2^{-i} \text{ for all } i \geq N.$$

Therefore, the binary representation of $a^{(i)}$ and b will be the same as $i \to \infty$.

Let $D: 2^{\mathbb{N}_+} \times 2^{\mathbb{N}_+} \to [0, \infty)$ be defined by

$$D(a,b) = \sup_{n \in \mathbb{N}_+} d'(a_n, b_n) \text{ where } d' \text{ is the discrete metric on } \{0,1\}.$$

Let $D': 2^{\mathbb{N}_+} \times 2^{\mathbb{N}_+} \to [0, \infty)$ be defined by

$$D'(a,b) = \sum_{1}^{\infty} 2^{-n} \frac{d'(a_n, b_n)}{1+d'(a_n,b_n)}.$$

Note that if $D(a^{(i)}, b) \to 0$ or $D'(a^{(i)}, b) \to 0$ as $i \to \infty$, then $d'(a_n^{(i)}, b_n) \to 0$ as $i \to \infty$ for all $n \in \mathbb{N}_+$. Therefore, the convergence with respect to the ultrametric d is equivalent to the convergence with respect to D or D'. Can you conclude the equivalence of d and D (or equivalence of d and D')?