Boğaziçi University
Department of Mathematics
Math 331 Metric Spaces
Fall 2025 – First Midterm Exam

1	2	3	4	5	$\sum$
PROPOSED		SOLUTIONS			
20 pts	20 pts	20 pts	24 pts	18 pts	100 pts

Date: October 27, 2025	Full Name:
Time: 17:00-19:00	

In this exam, $X = (X, d)$ denotes an arbitrary metric space; $(V, ||\cdot||_V)$ denotes an arbitrary normed space. A metric g on a set Y that satisfies the stronger axiom $g(a, c) \leq \max(g(a, b), g(b, c))$ for all $a, b, c \in Y$ is called an ultrametric on Y.

Recall the expression for the *p-norm* over $\mathbb{R}^n$: $||\mathbf{x}||_p \overset{\text{def}}{=} \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}$.

1. (a) (p. 5) For any $x, y \in X$, let $d'(x, y) = \min(d(x, y), 1)$. Show that this *bounded* d' is also a metric on X.
   (b) (p. 74) Show that (X, d') is homeomorphic to (X, d) when (X, d) is a bounded space.

(a) • $d'(x,y)=0 \iff d(x,y)=0 \iff x=y$. • $d'(x,y)=d'(y,x)$.

• $d'(x,y) \leq \min\left(d(x,z)+d(y,z),\ 1\right) \underset{\text{needs justification}}{\leq} \min\left(d(x,z),1\right) + \min\left(d(y,z),1\right) = d'(x,z)+d'(y,z)$

(b) d and d' are equivalent:

$\forall x,y \in X: d'(x,y) \leqslant d(x,y) \leqslant M \cdot d'(x,y)$ where M is a bound for d over X.

By thm, the result follows.

Note. Claim true even if (X,d) is not bounded: $\forall B^d$ is open wrt d' & $\forall B^{d'}$ is open wrt d. Can you show this?

2. Prove directly from definitions:
   (p. 65) All linear mappings $T : (\mathbb{R}^n, ||\cdot||_1) \to V$ are continuous.

Fix a basis $\{u_1, \dots, u_n\}$ for $\mathbb{R}^n$. Given T, set $M = \max_j Tu_j$.

For $u \in \mathbb{R}^n$, express $u = \sum a_j u_j$. For continuity at $v = \sum b_j u_j$, given $\varepsilon > 0$, we want:

$\varepsilon > \|Tu - Tv\| = \sum |a_j - b_j| \cdot T(u_j) \leqslant M \sum_j |a_j - b_j| = M \cdot \|u - v\|_1$

Note that without the red part, we already proved T is Lipschitz so T is cont.

So just choose $\delta = \varepsilon / M$. Then $\|u\|_1 < \delta \Rightarrow \|Tu\|_V \leqslant M \cdot \|u\|_1 < \varepsilon$.

No "big thms" are allowed here. Just use the basic definitions.

3. Consider the space $\mathcal{C}$ of all continuous functions from $[0, 1]$ to $[-1, +1]$ with the ∞-norm (the supremum norm). Show that the set $P = \{f \in \mathcal{C} : |f(x)| > 0 \text{ for all } x \in [0, 1]\}$ is open in $\mathcal{C}$.

Let $g \in P$. Set $s = \inf_{x \in [0,1]} |g(x)| = \min |g(x)|$. Note $s > 0$.

claim: $B_{s/2}(g) \subset P$. pf. Let $h \in B$. Then $\forall x \in [0,1]$, $|g(x)| \leqslant |g(x) - h(x)| + |h(x)|$

$< \frac{s}{2} + |h(x)|$

$\Rightarrow \forall x,\ |h(x)| > |g(x)| - s/2 \geqslant s - \frac{s}{2} = s/2$

So $h \in C$.

For this page do not use any other paper for solutions. Use the spaces provided below.

4. TRUE or FALSE. 8 pts each... Either prove or refute. Refuting is a proof; you can do this by giving an explicit counterexample and proving that that example works.

(a) Every function from X to a discrete metric space is continuous.

FALSE: Let $X=\mathbb{R}$, Y a discrete space with $a,b\in Y$.

Consider $f:\mathbb{R}\to Y$, $f(x)=\begin{cases} a, & x\in\mathbb{Q} \\ b, & x\notin\mathbb{Q}\end{cases}$. OR easier: $id:(\mathbb{R},\text{eucl})\to(\mathbb{R},\text{discrete})$

Observe $\forall\delta>0$, $f(B_\delta(0))=\{a,b\}$.

So given $0<\varepsilon<1$, there is no $\delta>0$ s.t. $f(B_\delta(0))\subset B_\varepsilon(a)=a$

(b) $||\mathbf{x}||_{1/2}$ is a norm on $\mathbb{R}^n$, $n>0$.

FALSE: For $n=2$ and $(1,0),(0,1)\in\mathbb{R}^2$

$\|(1,0)+(0,1)\|_{1/2} = \|(1,1)\|_{1/2} = (\sqrt{1}+\sqrt{1})^2 = 4$,

while

$\|(1,0)\|_{1/2}+\|(0,1)\|_{1/2} = 1+1 = 2$

triangle inequality fails

(c) Let B be an arbitrary open ball in a space Y with an ultrametric g. Then any point of B is a center of B.

TRUE: Let $B=B_r(y)=\{x\in Y \mid d(x,y)<r\}$

Take $z\in B$. Then $\forall x\in B$, $d(x,z)\leq\max(g(x,y),g(y,z))$
$<\max(r,r)=r$.

Hence $B_r(y)\subseteq B_r(x)$. Similarly $B_r(x)\subseteq B_r(y)$.

5. TRUE or FALSE? 3 pts each... No justification required. An incorrect answer cancels a correct one.

1. $\boxed{F}$ In any metric space, any finite subset has empty interior.
In a discrete space, the interior of a singleton is nonempty.
2. $\boxed{T}$ For any $a\neq b\in X$, there are **open** sets A and B in X such that $a\in A$, $b\in B$, $A\cap B=\emptyset$.
Set $r=d(a,b)$. Then the open balls $B_{r/2}(a)$ and $B_{r/2}(b)$ are disjoint.
3. $\boxed{F}$ For the norm-∞ unit sphere in $\mathbb{R}^2$, its diameter in any p-norm is 2.
$d_1(A,B)=4$ in the 1-norm.

4. $\boxed{T}$ With respect to any p-norm on $\mathbb{R}^2$, the sequence $\left((\frac{1}{j},\frac{1}{j^2})\right)_{j=1}^{\infty}$ converges to $(0,0)$.
The sequence converges to 0 in Euclidean norm. Hence in any p-norm.
5. $\boxed{F}$ Let d and d' be **equivalent** metrics on X. More carefully: $(\mathbb{R},\text{eucl})$ & $(\mathbb{R},d)$ with $d(x,y)=|\arctan x-\arctan y|$
A sequence is Cauchy with respect to d **if and only if** it is Cauchy with respect to d'.
$((0,1),\text{eucl})$ homeom to $(\mathbb{R},\text{eucl})$. However $(1-\frac{1}{n})_{n=1}^{\infty}$ is Cauchy in the 1st, not in 2nd.
6. $\boxed{T}$ A **linear** mapping from one normed space to another is continuous **if and only if** it is bounded on bounded sets.
a thm