Boğaziçi University
Department of Mathematics
Math 331 Metric Spaces
Fall 2025 – Second Midterm Exam.

1	2	3	4	$\sum$
PROPOSED	SOLUTIONS			
25 pts	25 pts	25 pts	25 pts	100 pts

Date: December 1, 2025 Time: 17:00-19:00	Full Name:

In this exam, $X = (X, d)$ and $Y = (Y, \tau)$ denote metric spaces; $A \subsetneq X$ is a subspace of X. $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ are Banach spaces. Any vector space in this exam is over $\mathbb{R}$.
$\rightsquigarrow$ means "write here".

You can use any theorem that was proven in my classes. Moreover I encourage you use them instead of proving many things from scratch.

1. Let A be sequentially compact in X.
   (a) [2] Give the definition of sequential compactness:
   $\rightsquigarrow$ A is sequentially compact if every sequence in A has a convergent subsequence in A.

   (b) [3] Fill in the blanks: if $g : P \to S$ is continuous and P is compact, then so is $g(P)$.
   (c) [20] (p. 129) Let $x \in X - A$. Show that there exists a point $a \in A$ such that $d(x, a) = d(x, A)$.
   Recall: $d(x, A) \doteq \inf\{d(x, a) | a \in A\}$. Hint: The quickest proof is through part (b).

Recall $d : X \times X \longrightarrow \mathbb{R}$ is continuous. Fix $x \in X - A$. Since $\{x\} \times A$ is compact, $d(\{x\} \times A)$ is compact. Hence d attains its minimum at some point $(x, a) \in \{x\} \times A$.

2. (p. 143) In an arbitrary metric space, if you don't have a nice theorem like Heine-Borel, it is in general difficult to find compact subspaces. Here is a famous compact subspace of ℓ^2.

(a) [4] Give the definition of the space ℓ^2 and the associated norm $||\cdot||_2$:

$\ell^2 =$ the set of real sequences $a=(a_n)$ with $\sum_{n=1}^{\infty} a_n^2$ finite.

$||a||_2 = \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2}$

The Hilbert cube $K \subset \ell^2$ is the set of all sequences $a = (a_n)_{n=1}^{\infty} \in \ell^2$ such that $0 \le a_n \le 1/n$ for each n. We show K is compact by showing that K is (c) complete, and (d) totally bounded.

(b) [10] Show that for any $a \in K^c$, there is $r > 0$ such that $B_r(a) \subset K^c$. (Recall $\forall b \in \ell^2, \forall m, |b_m| \le ||b||_2$.) This proves K is closed in ℓ^2.

$a \in K^c \Leftrightarrow a \in \ell^2$ and $\exists m : a_m > \frac{1}{m}$ (or $a_m < 0$). Then set $r = (a_m - \frac{1}{m})/2$ (or $r = \frac{|a_m|}{2}$).

In that case $B_r(a) \subset K^c$, because $\forall b = (b_n) \in B_r(a)$,

$|b_m - a_m| \le \|b - a\|_2 < r \Rightarrow b_m > \frac{1}{m}$ (or $b_m < 0$) $\Rightarrow b \notin K$.

(c) [5] How does it follow from part (b) that K is complete?

ℓ^2 complete $K \subset \ell^2$ closed $\rightarrow K$ complete

(d) [6] Show that K, is totally bounded. Hint: ~~If~~ you delete the tails of sequences, you can cover what remains by finitely many ε-balls. Why? After this, how would you proceed?

$\forall \varepsilon > 0$ given, $\exists N$ s.t. $\sum_{k=N+1}^{\infty} a_k^2 \le \sum_{k=N+1}^{\infty} \frac{1}{k^2} < \varepsilon/3$, since $\sum \frac{1}{k^2}$ is convergent.

Define $K_N = \{(a_1, \dots, a_N, 0, 0, \dots) \in K\}$. Since $K_N \subset \mathbb{R}^N$ & is bounded, it is totally bounded. Cover it by finitely many $\frac{\varepsilon}{3}$-balls $B_{\varepsilon/3}(u_i)$.

Now for any $a \in K$, $(a_1, \dots, a_N, 0, 0, \dots)$ lies in some $B_{\varepsilon/3}(u_i)$ then

$\|a - u_i\|_2 = \frac{\varepsilon}{3} + \sum_{N+1}^{\infty} \frac{1}{k^2} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$, that is, $a \in B_\varepsilon(u_i)$.

This proves that given $\varepsilon > 0$, K can be covered by finitely many ε-balls.

3. Suppose X is compact.

(a) [2] Give the definition of the sup norm on the space $C(X, \mathbb{R})$:

For $f \in C(X, \mathbb{R})$, $\|f\|_\infty = \sup_{x \in X} |f(x)|$

(b) [5] Tell explicitly the product metric μ on the product space $C(X, \mathbb{R}) \times X$:

(Use this metric in part (c). If you cannot answer (a)&(b) correctly, it will be impossible to get any points in part (c)!)

$\mu((f, x), (g, y)) = \max(\|f - g\|_\infty, d_X(x, y))$

(c) [18] (p. 161) Consider the function $v : C(X, \mathbb{R}) \times X \to \mathbb{R}$, $v(f, x) = f(x)$. Show that v is continuous.

Hint: Given $\alpha \in \mathbb{R}$ and $v(f, x) = \alpha$; when $\varepsilon > 0$ is given, find an open ball B with center (f, x) such that $v(B)$ lies in $B_\varepsilon(\alpha)$.

$\forall f$, given $\varepsilon > 0$, $\exists\, \delta_{\varepsilon,f} > 0$ s.t. $\forall x, y \in X$, $d_X(x, y) < \delta_{\varepsilon,f} \Rightarrow |f(x) - f(y)| < \varepsilon/2$

Now given $\alpha \in \mathbb{R}$ with $v(f, x) = f(x) = \alpha$, and given $\varepsilon > 0$,

Set $U_\delta = \{(g, y) \in C(X, \mathbb{R}) \times X \mid \|g - f\|_\infty < \varepsilon/2 \ \& \ d_X(y, x) < \delta_{\varepsilon,f}\}$, open in $C(X) \times X$.

Then $\forall (g, y) \in U_\delta$, $|v(g, y) - v(f, x)| = |g(y) - f(x)| \leq |g(y) - f(y)| + |f(y) - f(x)|$

$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$

This proves $v(U_\delta) \subset B_\varepsilon(f(x))$.

4. (a) [3] Give the definition of uniform continuity for a function $g : X \to Y$:

$\forall \varepsilon > 0, \exists \delta_\varepsilon : \forall x, y \in X \quad d_X(x,y) < \delta_\varepsilon \Rightarrow d_Y(g(x), g(y)) < \varepsilon.$

(b) [6] (p. 168) Let $h : V \to W$ be linear and continuous. Show that h is uniformly continuous.

Let $D \subset V$ be a dense vector subspace of V; $f : D \to W$ be linear and continuous (and hence uniformly continuous by (b)). We have proven that such an f has a unique continuous extension $F : V \to W$.

(c) [4] Remind me how the extension F is defined at a point $x \in V - D$:

Take any (a_n) in D with limit x. Define $F(x) = \lim_{n\to\infty} f(a_n)$.

(d) [6] (p. 168) Show that F is linear too.

(e) [6] (p. 168) If for all $x \in D$, $||f(x)||_W \leq K \cdot ||x||_V$ then prove that for all $x \in V$, $||F(x)||_W \leq K \cdot ||x||_V$. (This implies that f and its unique extension F have the same operator norm.)

(b) Recall linear $h: V \to W$ is continuous iff $\exists K > 0$ s.t. $\|h(x)\|_W < K \cdot \|x\|_V, \forall x \in V$.

Given $\varepsilon > 0$, let $\delta_\varepsilon = \varepsilon / K$. Then $\forall x, y \in V$,

$$\|x-y\|_V < \delta_\varepsilon \Rightarrow \|h(x)-h(y)\|_W < K \cdot \|x-y\|_V < K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

(d) Let $x, y \in V$; $(a_n), (b_n)$ in D with $\lim a_n = x$, $\lim b_n = y$. Let $\alpha \in \mathbb{R}$.

Observe $\alpha x + y = \lim_{n\to\infty} (\alpha a_n + b_n)$. Then

$$\text{Then } F(\alpha x + y) = \lim_{n\to\infty} f(\alpha a_n + b_n) \overset{f \text{ is linear}}{=} \lim_{n\to\infty} \alpha f(a_n) + f(b_n) = \alpha F(x) + F(y).$$

(e) $\forall x \in V$, $$\|F(x)\|_W = \left\|\lim_{n\to\infty} f(a_n)\right\|_W \overset{\text{norm is contin.}}{=} \lim_{n\to\infty} \|f(a_n)\|_W \leq K \cdot \lim_{n\to\infty} \|a_n\|_V$$

$$= K \cdot \|\lim a_n\|_V = K \cdot \|x\|_V.$$