Math 331
Quiz 6 Solution

Problem

If $a \neq b$, $a, b \in \mathbb{B}$ and $s = \dfrac{a+b}{2} \in \mathbb{B}$ then $\dfrac{a+b}{2} = \dfrac{k}{3^n}$ for some integers k, n where $3 \nmid k$, and $\mathbb{B}$ is the Cantor's Middle Thirds Set.

Solution. Since $a, b \in \mathbb{B}$, we have

$$a = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \quad b = \sum_{i=1}^{\infty} \frac{b_i}{3^i} \tag{1}$$

where $a_i, b_i \in \{0, 2\}$ for all i. Also, for s, we have

$$s = \sum_{i=1}^{\infty} \frac{a_i + b_i}{2 \cdot 3^i} = \sum_{i=1}^{\infty} \frac{c_i}{3^i} \tag{2}$$

where $c_i = \dfrac{a_i + b_i}{2}$. Since $a_i, b_i \in \{0, 2\}$, possible values for c_i are $0, 1, 2$. Now, if $c_i \neq 1$ for all i, then we have that either $a_i = b_i = 2$ or $a_i = b_i = 0$ for all i, thus $a = b$, contradicting our assumption. So, we have $c_m = 1$ for some m. But, since $s \in \mathbb{B}$ this can only happen when

$$\begin{aligned} & s = (0.c_1 \dots c_{m-1}1\bar{2})_3 = (0.c_1 \dots c_{m-1}2\bar{0})_3 \\ \text{or} \quad & s = (0.c_1 \dots c_{m-1}1\bar{0})_3 = (0.c_1 \dots c_{m-1}0\bar{2})_3 \end{aligned} \tag{3}$$

This is because ternary representations are unique except for when they end with $\bar{0}$ or $\bar{2}$. So, we have

$$\begin{aligned} & s = \frac{c_1}{3} + \frac{c_2}{3^2} + \dots + \frac{c_{m-1}}{3^{m-1}} + \frac{2}{3^m} = \frac{A}{3^m} \\ \text{or} \quad & s = \frac{c_1}{3} + \frac{c_2}{3^2} + \dots + \frac{c_{m-1}}{3^{m-1}} + \frac{1}{3^m} = \frac{A}{3^m} \end{aligned} \tag{4}$$

for some integer A, if we factor out powers of 3 from A, we obtain the result, $s = k/3^n$ (in fact, $3 \nmid A$, but doesn't matter).

$\square$