

Theorem 3.2.1 (Existence and Uniqueness Theorem)

Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad (4)$$

where p , q , and g are continuous on an open interval I that contains the point t_0 . Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I .

We emphasize that the theorem says three things:

1. The initial value problem *has* a solution; in other words, a solution *exists*.
2. The initial value problem has *only one* solution; that is, the solution is *unique*.
3. The solution ϕ is defined *throughout the interval* I where the coefficients are continuous and is at least twice differentiable there.

In each of Problems 7 through 12, determine the longest interval in which the given initial value problem is certain to have a unique twice-differentiable solution. Do not attempt to find the solution.

3. 2. 8. $(t-1)y'' - 3ty' + 4y = \sin t, \quad y(-2) = 2, \quad y'(-2) = 1$

12. $(x-2)y'' + y' + (x-2)(\tan x)y = 0, \quad y(3) = 1, \quad y'(3) = 2$

$$8) \quad y'' - \underbrace{\frac{3t}{t-1}}_{p(t)} y' + \underbrace{\frac{4}{t-1}}_{q(t)} y = \underbrace{\frac{\sin t}{t-1}}_{g(t)}$$

$$t_0 = -2$$

discts at $t=1$

$$I = (-\infty, 1) \cup (1, \infty)$$

by Ex & Un. Thm.

$$12) \quad y'' + \underbrace{\frac{1}{x-2}}_{p(x)} y' + \underbrace{(\tan x)}_{q(x)} y = 0$$

$$t_0 = 3 \Rightarrow I = (2, \frac{3\pi}{2})$$

discts at $x=2$

discts at $x = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}$

$$\dots \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, 2\right) \cup \left(2, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \dots$$

We talked about Wronskian and reviewed thms

3.2.2, 3.2.3, 3.2.4 and their proofs. (Read p. 148-150 on 10th Ed.)

18. If the Wronskian W of f and g is $t^2 e^t$, and if $f(t) = t$, find $g(t)$.

$$W(f, g)(t) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = (f g' - g f')(t) = t^2 e^t$$

$$f(t) = t \Rightarrow t g' - g = t^2 e^t$$

$$\mu(t) = c \cdot \frac{1}{t}$$

1st order, linear ODE

$$g(t) = t e^t + c t$$

$\Rightarrow$ integrating factor $\mu(t)$

In each of Problems 24 through 27, verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions? $\{y_1(t), y_2(t)\}$

3.2. 24. $y'' + 4y = 0$; $y_1(t) = \cos 2t$, $y_2(t) = \sin 2t$ $y(0) = 1$ $y'(0) = -1$

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2 \sin 2t & 2 \cos 2t \end{vmatrix} = 2(\cos^2 2t + \sin^2 2t) = 2 e^t$$

YES.

discussion:

$$W(y_1, y_2)(t) = \begin{vmatrix} \dots & \dots \\ \dots & \dots \end{vmatrix} = \dots = t \rightarrow \text{for } t_0 = 1 \quad W = 1$$

YES.

$W(t) \equiv 0$

$$= t - t = 0 \rightarrow \text{No.}$$

$$y'' + 4y = 0 \quad y_1 = \cos 2t \quad y_2 = \sin 2t$$

$$y = c_1 \cos 2t + c_2 \sin 2t$$

$$y' = c_1 (-2 \sin 2t) + c_2 (2 \cos 2t)$$

$$y(0) = 1 \Rightarrow c_1 \cos(0) + c_2 \sin(0) = 1$$

$$y'(0) = -1 \Rightarrow c_1 (-2 \sin(0)) + c_2 (2 \cos(0)) = -1$$

$$\begin{bmatrix} \cos(0) & \sin(0) \\ -2 \sin(0) & 2 \cos(0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

After Abel's Thm we see, if such a Wronskian exists, then all points to where $W(t_0) = 0$ are excluded somehow. (for example, $p(x)$ is undefined for those t_0 's etc.)

Theorem 3.2.7 (Abel's Theorem)⁴

If y_1 and y_2 are solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad (22)$$

where p and q are continuous on an open interval I , then the Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = c \exp \left[- \int p(t) dt \right], \quad (23)$$

where c is a certain constant that depends on y_1 and y_2 , but not on t . Further, $W(y_1, y_2)(t)$ either is zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

Recall
the E2un.
Thm
3.2.1!

In each of Problems 29 through 32, find the Wronskian of two solutions of the given differential equation without solving the equation.

3.2.30. $(\cos t)y'' + (\sin t)y' - ty = 0$

Abel

$$W(y_1, y_2)(t) = c e^{-\int p(t) dt} = c e^{-\int \frac{\sin t}{\cos t} dt} = c e^{\ln |\cos t| + c} = c (\cos t) = 0 \text{ if } \cos t = 0$$

undefined if $\cos t = 0$

3.2.34. If the differential equation $ty'' + 2y' + te^t y = 0$ has y_1 and y_2 as a fundamental set of solutions and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$.

Abel

$$W(y_1, y_2)(t) = c e^{-\int p(t) dt} = c e^{-\int \frac{2}{t} dt} = c e^{-2 \ln |t| + c} = c e^{\ln t^2} = c t^2$$
$$W(1) = 2 \Rightarrow c(1)^2 = 2 \Rightarrow c = 2$$
$$W(5) = ? = 2 \cdot (5)^2 = 2/25.$$

In each of Problems 7 through 16, find the general solution of the given differential equation.

3.3. 7. $y'' - 2y' + 2y = 0$

2nd order, linear ODE @ hom. w/ const. coeffs.

char eqn: $r^2 - 2r + 2 = 0 \Rightarrow r_{1,2} = 1 \pm i \rightarrow y_1 = e^t \sin t$
 $(r-1)^2 + 1 = 0 \rightarrow y_2 = e^t \cos t$

$y = e^{rt}$

$$e^{(1+i)t} = e^t \cdot e^{it} = e^t (\cos t + i \sin t)$$

$$e^{(1-i)t} = e^t \cdot e^{-it} = e^t (\cos(-t) + i \sin(-t)) = e^t (\cos t - i \sin t)$$

Thm 3.2.6

$y = u(t) + i v(t)$
 $\Rightarrow u(t), v(t)$ solve

$$\underline{e^{ait} = \cos at + i \sin at = cis at}$$

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{ait} = \sum_{n=0}^{\infty} \frac{(ait)^n}{n!} = \sum_{n=0}^{\infty} \frac{(ai)^n t^n}{n!} \\ &= \sum_{k=0}^{\infty} (i)^{2k} \frac{(at)^{2k}}{(2k)!} + \underbrace{\sum_{m=0}^{\infty} (i)^{2m+1} \frac{(at)^{2m+1}}{(2m+1)!}}_{i \left(\right)} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(at)^{2k}}{(2k)!} + i \sum_{m=0}^{\infty} (-1)^m \frac{(at)^{2m+1}}{(2m+1)!} \\ &= \cos at + i \sin at \end{aligned}$$

$$\begin{aligned} W(e^t \cos t, e^t \sin t) &= \begin{vmatrix} e^t \cos t & e^t \sin t \\ e^t \cos t - e^t \sin t & e^t \sin t + e^t \cos t \end{vmatrix} \\ &= e^{2t} \begin{vmatrix} \cos t & \sin t \\ \cos t - \sin t & \sin t + \cos t \end{vmatrix} \\ &= e^{2t} (\cancel{\cos t \sin t} + \cos^2 t - \cancel{\cos t \sin t} + \sin^2 t) = e^{2t} \end{aligned}$$

gen. soln:

$$y = c_1 y_1 + c_2 y_2 = c_1 e^t \cos t + c_2 e^t \sin t$$

$$r = \underline{a \pm ib} \rightarrow y_1 = \underline{e^{at}} \underline{\cos bt} \quad y_2 = \underline{e^{at}} \underline{\sin bt}$$

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} e^{at} \cos bt & e^{at} \sin bt \\ ae^{at} \cos bt - e^{at} b \sin bt & ae^{at} \sin bt + e^{at} b \cos bt \end{vmatrix} \\ &= e^{2at} \begin{vmatrix} \cos bt & \sin bt \\ a \cos bt - b \sin bt & a \sin bt + b \cos bt \end{vmatrix} \end{aligned}$$

$$= e^{2at} (a \cos bt \sin bt + b \cos^2 bt - a \cos bt \sin bt + b \sin^2 bt)$$

$$= e^{2at} \quad b \neq 0 \quad \rightarrow \quad \text{so } \{e^{at} \cos bt, e^{at} \sin bt\} \text{ always forms a fundamental set of solutions unless } b=0 \text{ of course!!}$$

In each of Problems 17 through 22, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t .

3.3.17. $y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$
 $r^2 + 4 = 0$

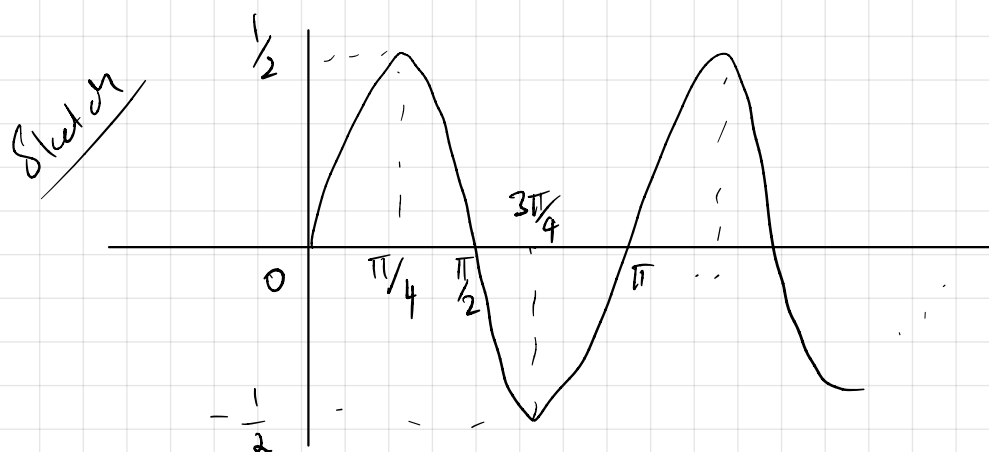
2nd order, lin. ODE, hom & const. coef. $\Rightarrow$ char. eqn.
 $r^2 = -4 \Rightarrow r_{1,2} = \pm 2i \rightarrow y_1 = \cos 2t$
 $\rightarrow y_2 = \sin 2t$

$y = c_1 \cos 2t + c_2 \sin 2t \rightarrow$ general soln.

$y' = -2c_1 \sin 2t + 2c_2 \cos 2t$

IVs: $y(0) = 0 \Rightarrow c_1 = 0$
 $y'(0) = 1 \Rightarrow 2c_2 = 1 \Rightarrow c_2 = 1/2$

particular soln: $y = \frac{1}{2} \sin 2t$



behavior
 $t \uparrow$
 oscillates between
 $-\frac{1}{2}$ & $\frac{1}{2}$.

In each of Problems 1 through 10, find the general solution of the given differential equation.

3.4. 6. $y'' - 6y' + 9y = 0$

char eqn: $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0 \rightarrow r_1, r_2 = 3 \begin{cases} \rightarrow y_1 = e^{3t} \\ \rightarrow y_2 = te^{3t} \end{cases}$

general soln: $y = c_1 e^{3t} + c_2 te^{3t}$

How we find $y_2 = te^{3t}$?

We know $c_1 y_1$ is a solution. Instead of c_1 we choose to use a func. $v(t)$, and assume $v(t)y_1(t)$ solves the DE. ("Reduction of order")

$y = v(t)e^{3t} \quad y' = v'(t)e^{3t} + 3v(t)e^{3t}$

$y'' = v''(t)e^{3t} + 3v'(t)e^{3t} + 3v'(t)e^{3t} + 9v(t)e^{3t}$

$0 = y'' - 6y' + 9y = e^{3t} [v'' + 6v' + 9v - 6v' - 18v + 9v]$

$0 = v'' \rightarrow v' = c_1 \rightarrow v = ct + c_2$

$y = v \cdot y_1 = (c_1 t + c_2) e^{3t} = c_1 te^{3t} + c_2 e^{3t} \quad y = c_1 y_1 + c_2 y_2$

$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & e^{3t} + 3te^{3t} \end{vmatrix} = e^{6t} (1 + 3t - 3t) = e^{6t}$

2nd order homogen.

Reduction of order works for linear DE with a known soln.

$y = v \cdot y_1$

$y' = v' \cdot y_1 + v y_1'$

$y'' = v'' y_1 + 2v' y_1' + v y_1''$

$y'' + p(t)y' + q(t)y = 0$, y_1 soln $\Rightarrow y_1'' + p y_1' + q y_1 = 0$

$v'' y_1 + 2v' y_1' + v y_1'' + p(v' y_1 + v y_1') + q v y_1 = 0$

$v'' \cdot y_1 + v' (y_1'' + p y_1' + q y_1) = 0$

$$0 = y'' - 6y' + 9y = y'' - 3y' - 3y' + 9y$$

$$= \underbrace{(y'' - 3y')}_{u'(t)} - 3\underbrace{(y' - 3y)}_{u(t)}$$

$$0 = u' - 3u$$

$$\xrightarrow{c \neq 0} 3u = u' \xrightarrow{c=0} u(t) = ce^{3t}$$

$$y' - 3y = \underbrace{ce^{3t}}_{te^{3t}}$$

$$y' - 3y = 0 \rightarrow y = e^{3t}$$

(use $\mu(x)$ & solve it).

In each of Problems 11 through 14, solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t .

3.4. 12. $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$

general soln: $y = c_1 e^{3t} + c_2 t e^{3t}$

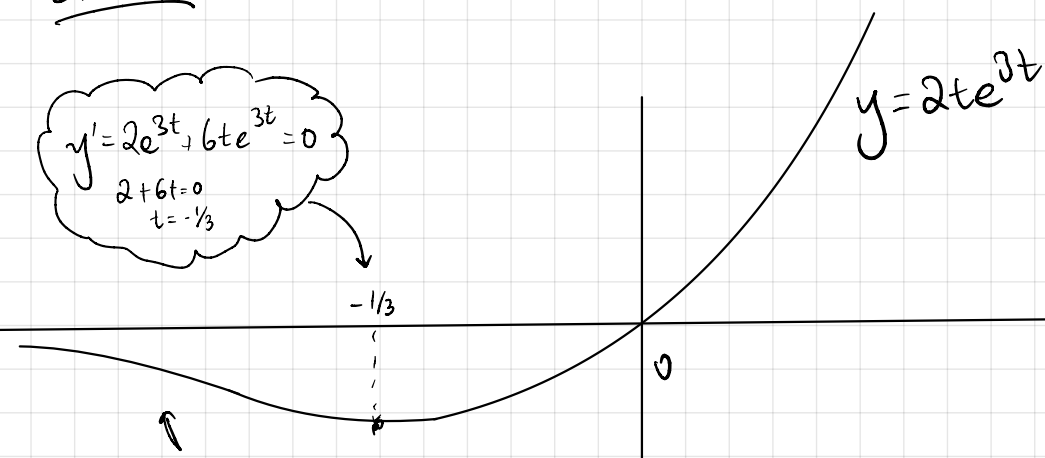
I.V.s: $y(0) = 0 \rightarrow y(0) = c_1 = 0 \rightarrow y = c_2 t e^{3t}$

$$y'(0) = 2 \rightarrow y'(t) = c_2 e^{3t} + 3c_2 t e^{3t}$$

$$y'(0) = c_2 = 2$$

part. soln: $y = 2t e^{3t}$

Sketch



Behavior

$$\lim_{t \rightarrow \infty} y = \infty$$

y increases exponentially as $t \uparrow$.

Since $\lim_{t \rightarrow -\infty} y = 0$ (write the limit & use L'Hopital if you don't see it immediately) but $y < 0$ for $t < 0$.

3.4.18. Consider the initial value problem

$$9y'' + 12y' + 4y = 0, \quad y(0) = \underline{a} > 0, \quad y'(0) = -1.$$

(a) Solve the initial value problem.

(b) Find the critical value of a that separates solutions that become negative from those that are always positive.

(a) 2nd order lin. DE w const. coef.

after some t_0 , $\forall t > t_0$
 $y(t) < 0$.

char eqn: $3r^2 + 12r + 4 = 0$

$$(3r + 2)^2 = 0 \rightarrow r_{1,2} = -\frac{2}{3}$$

gen. soln: $y = c_1 e^{-\frac{2}{3}t} + c_2 t e^{-\frac{2}{3}t}$

$$y_1 = e^{-2t/3} \quad y_2 = t e^{-2t/3}$$

IVs: $y(0) = a \rightarrow y(0) = c_1 = a > 0$.

$$y = a e^{-\frac{2}{3}t} + c_2 t e^{-\frac{2}{3}t}$$

$$y' = -\frac{2}{3} a e^{-\frac{2}{3}t} + c_2 e^{-\frac{2}{3}t} - \frac{2}{3} c_2 t e^{-\frac{2}{3}t}$$

$$= e^{-\frac{2}{3}t} \left(-\frac{2}{3} a + c_2 \right) - \frac{2}{3} c_2 t e^{-\frac{2}{3}t}$$

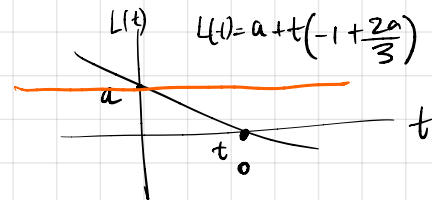
$$y'(0) = -1 \rightarrow y'(0) = -\frac{2}{3} a + c_2 = -1$$

$$\Rightarrow c_2 = -1 + \frac{2a}{3}$$

IVs soln: $y = a e^{-\frac{2}{3}t} + \left(-1 + \frac{2a}{3} \right) t e^{-\frac{2}{3}t}$

(b) $\lim_{t \rightarrow \infty} \underbrace{a e^{-\frac{2}{3}t}}_0 + \left(-1 + \frac{2a}{3} \right) \underbrace{t e^{-\frac{2}{3}t}}_0 = 0$ nothing interesting.

$$y = \underbrace{e^{-\frac{2}{3}t}}_{>0} \left[\underbrace{a + t\left(-1 + \frac{2a}{3}\right)}_{>0} \right]$$



as $t \rightarrow \infty$ $y < 0$ if $-1 + \frac{2a}{3} < 0 \rightarrow a < \frac{3}{2}$
 $y > 0$ if $-1 + \frac{2a}{3} \geq 0 \rightarrow a \geq \frac{3}{2}$

critical $a = \frac{3}{2}$.

In each of Problems 23 through 30, use the method of reduction of order to find a second solution of the given differential equation.

3.4. 25. $t^2 y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$

$$y = \underline{v(t)} t^{-1}$$

$$y = v(t) y_1$$

$$y' = \frac{v'}{t} - \underline{\frac{v}{t^2}}$$

$$y'' = \frac{v''}{t} - \frac{v'}{t^2} - \frac{v'}{t^2} + \underline{\frac{2v}{t^3}}$$

$$t^2 \left(\frac{v''}{t} - \frac{v'}{t^2} - \frac{v'}{t^2} + \frac{2v}{t^3} \right) + 3t \left(\frac{v'}{t} - \frac{v}{t^2} \right) + \frac{v}{t} = 0$$

$$tv'' - 2v' + \cancel{2\frac{v}{t}} + 3v' - \cancel{3\frac{v}{t}} + \frac{v}{t} = 0$$

$$\begin{aligned} v' &= u \\ v'' &= u' \end{aligned}$$

(1) $tv'' + v' = 0$
 $t u' + u = 0$

1st order DE of v'

$$\int \frac{v''}{v'} dt = \int \frac{-1}{t} dt \rightarrow \ln|v'| = \ln|t|^{-1} + c$$

$|t| = t$ since $t > 0$.

$$v' = c t^{-1}$$

$e^c \mapsto c$

(2) $v' = c t^{-1}$

general soln:

$$v = c_1 \ln t + c_2 \rightarrow y = \frac{v}{t} = c_1 t^{-1} \ln t + c_2 t^{-1}$$