

3.5. Undetermined Coefficients

In each of Problems 1 through 14, find the general solution of the given differential equation.

3.5. 7. $y'' + 9y = t^2 e^{3t} + 6$

$y'' + 9y = t^2 e^{3t} \rightarrow y_2$ $y'' + 9y = 6 \rightarrow y_1$

hom. DE: $y'' + 9y = 0$ 2nd order lin, const coef.

char eqn: $r^2 + 9 = 0 \rightarrow r_{1,2} = \pm 3i$ $\rightarrow y_1 = \cos 3t$
 $\rightarrow y_2 = \sin 3t$

hom. soln: $y = c_1 \cos 3t + c_2 \sin 3t$

(1) nonhom: $y'' + 9y = 6$ "P(t) of deg 0"

$9c = 6 \Rightarrow c = 2/3$

meth. of undet. coeffs.

$y_1 = P(t)$ of deg 0. t^s
 $= c$
 $y_1' = 0$

s is the number of times 0 is a root of char. eqn.

($s=0$ here)

$y_1'' + 9y_1 = 6 \rightarrow$ soln of 1st part
 $+ y_2'' + 9y_2 = t^2 e^{3t} \rightarrow$ soln of 2nd part

 $(y_1 + y_2)'' + 9(y_1 + y_2) = t^2 e^{3t} + 6 \rightarrow y_{\text{particular}} = y_1 + y_2$

(2) nonhom: $y'' + 9y = t^2 e^{3t} = \underset{\text{deg 2}}{P(t)} \cdot e^{3t} \cdot t^s$

s is the number of times 3 is a soln of char. eqn. ($s=0$, here.)

$y_2'' + 9y_2 = t^2 e^{3t}$

$y_2 = (At^2 + Bt + C) \cdot e^{3t}$

$y_2' = (2At + B)e^{3t} + 3(At^2 + Bt + C)e^{3t}$
 $= e^{3t}(3At^2 + (2A + 3B)t + B + 3C)$

$y_2'' = 3e^{3t}(3At^2 + (2A + 3B)t + B + 3C) + e^{3t}(6At + 2A + 3B)$

$t^2 e^{3t} = \underbrace{3e^{3t}(3At^2 + (2A + 3B)t + B + 3C)}_{=1 \text{ ①}} + \underbrace{e^{3t}(6At + 2A + 3B)}_{=0 \text{ ②}} + \underbrace{9(At^2 + Bt + C)e^{3t}}_{=0 \text{ ③}}$

$t^2 = (9A + 9A)t^2 + (6A + 9B + 6A + 9B)t + 3B + 9C + 2A + 3B + 9C$

$$\textcircled{1} 18A = 1 \rightarrow A = 1/18$$

$$\textcircled{2} 12A^2 + 18B^3 = 0$$

$$1/9 = -3B \rightarrow B = -1/27$$

$$\textcircled{3} -\frac{2}{9} + 18C + \frac{1}{9} = 0$$

$$18C = \frac{1}{9} \quad C = \frac{1}{162}$$

$$y_1 = \frac{2}{3}$$

$$y_2 = \left(\frac{t^2}{18} - \frac{t}{27} + \frac{1}{162} \right) e^{3t}$$

$$y_{\text{part}} = \frac{2}{3} + \left(\frac{t^2}{18} - \frac{t}{27} + \frac{1}{162} \right) e^{3t}$$

gen. soln: $y = y_{\text{hom}} + y_{\text{par}}$

$$= c_1 \cos 3t + c_2 \sin 3t + \frac{2}{3} + \left(\frac{t^2}{18} - \frac{t}{27} + \frac{1}{162} \right) e^{3t}$$

In each of Problems 15 through 20, find the solution of the given initial value problem.

3.5 -16. $y'' + 4y = t^2 + 3e^t$, $y(0) = 0$, $y'(0) = 2$

hom DE: $y'' + 4y = 0$

char. eqn: $r^2 + 4 = 0 \rightarrow r_{1,2} = \pm 2i$

$$\begin{aligned} &\nearrow y_1 = \cos 2t \\ &\searrow y_2 = \sin 2t \end{aligned}$$

$$y_{\text{hom}} = c_1 \cos 2t + c_2 \sin 2t$$

nonhom DE: $y'' + 4y = \underbrace{t^2} + \underbrace{3e^t}$

$$y = y_{\text{par}} = (At^2 + Bt + C) \overset{t^5}{\quad} + De^t \overset{t^9}{\quad}$$

$$y' = 2At + B + De^t$$

$$y'' = 2A + De^t$$

$$y'' + 4y = 2A + De^t + 4(At^2 + Bt + C) + 4De^t = t^2 + 3e^t$$

$$(4A)t^2 + (4B)t + (2A + 4C) + (5D)e^t = (1)t^2 + (3)e^t$$

(0)t + (0)

but $s=0$
since $r_{1,2} \neq 0$.

but
 $q=0$
since
 $r_{1,2} \neq \pm 1$

$$4A = 1 \rightarrow A = 1/4 \quad 4B = 0 \rightarrow B = 0 \quad \frac{1/2}{2A} + 4C = 0 \rightarrow C = -1/8$$

$$5D = 3 \rightarrow D = 3/5$$

$$y_{\text{part}} = Y = \frac{t^2}{4} - \frac{1}{8} + \frac{3}{5}e^t$$

gen. soln = $y = y_{\text{hom}} + y_{\text{part}}$

$$= c_1 \cos 2t + c_2 \sin 2t + \frac{t^2}{4} - \frac{1}{8} + \frac{3}{5}e^t$$

$$y' = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{t}{2} + \frac{3}{5}e^t$$

ivs: $y(0) = 0 \rightarrow c_1 - \frac{1}{8} + \frac{3}{5} = 0 \rightarrow c_1 = -\frac{19}{40}$

$$y'(0) = 2 \rightarrow 2c_2 + \frac{3}{5} = 2 \rightarrow c_2 = -\frac{7}{10}$$

$$y = -\frac{19}{40} \cos 2t - \frac{7}{10} \sin 2t + \frac{t^2}{4} - \frac{1}{8} + \frac{3}{5}e^t$$

Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

3.5. 27. $y'' + 3y' + 2y = \underline{e^t(t^2 + 1) \sin 2t} + \underline{3e^{-t} \cos t} + 4e^t$

is $1 \pm 2i$ a soln to char eqn?

is $-1 \pm i$ a soln of char eqn?

is 1 a soln of char eqn?

questions we need to ask to determine if there should be a t^5 factor. (Check the table 3.5.1 on p. 182)

char eqn $r^2 + 3r + 2 = 0 \quad (r+2)(r+1) = 0 \quad r_1 = -2, r_2 = -1$

$$y_1 = e^{-2t} \quad y_2 = e^{-t}$$

$$y_{\text{part}} = (At^2 + Bt + C)e^t \sin 2t + (Dt^2 + Et + F)e^t \cos 2t + G e^{-t} \cos t + H e^{-t} \sin t + J e^t$$

$$y = \underline{v_1 y_1} + \underline{v_2 y_2}, \text{ where } v_1 \& v_2 \text{ are fncs of } t.$$

$$y' = \underline{v_1' y_1} + \underline{v_1 y_1'} + \underline{v_2' y_2} + \underline{v_2 y_2'} \rightarrow \text{1st constraint } v_1' \& v_2'$$

$$y'' = \underline{v_1'' y_1} + \underline{v_1' y_1'} + \underline{v_1 y_1''} + \underline{v_2'' y_2} + \underline{v_2' y_2'} + \underline{v_2 y_2''}$$

$$t^2 y'' - 2y = t^2 (v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'') - 2(v_1 y_1 + v_2 y_2)$$

$$= t^2 (v_1' y_1' + v_2' y_2') = 3t^2 - 1 \rightarrow \text{2nd constraint}$$

$$t^2 y_1'' - 2y_1 = 0$$

$$t^2 v_1 y_1'' - 2v_1 y_1 = 0$$

$$4 \quad t^2 y_2'' - y_2 = 0$$

$$t^2 v_2 y_2'' - 2v_2 y_2 = 0$$

$$v_1' \& v_2'$$

In each of Problems 13 through 20, verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, g is an arbitrary continuous function.

36.13. $t^2 y'' - 2y = 3t^2 - 1, \quad t > 0; \quad y_1(t) = t^2, \quad y_2(t) = t^{-1}$

$$y = \underline{v_1 t^2} + \underline{v_2 t^{-1}}$$

$$\& \text{ constraint: } v_1' y_1 + v_2' y_2 = 0$$

$$y' = \underline{v_1' t^2} + \underline{v_2' t^{-1}} = 0$$

$$v_1' t^2 + v_2' t^{-1} = 0$$

$$+ v_1(2t) + v_2(-t^{-2})$$

$$y'' = 2v_1' t + \underline{2v_1} - v_2' t^{-2} + \underline{2v_2 t^{-3}}$$

$$t^2 y'' - 2y = t^2 (2v_1' t + \underline{2v_1} - v_2' t^{-2} + \underline{2v_2 t^{-3}}) - 2(\underline{v_1 t^2} + \underline{v_2 t^{-1}})$$

$$= 2v_1' t^3 - v_2' = 3t^2 - 1$$

$$(1) \quad v_1' t^2 + v_2' t^{-1} = 0$$

$$\cdot t^3 \rightarrow$$

$$v_1' t^3 + v_2' = 0$$

$$v_2' = -t^2 + \frac{1}{3}$$

$$(2) \quad 2v_1' t^3 - v_2' = 3t^2 - 1$$

$$+$$

$$3v_1' t^3 = 3t^2 - 1$$

$$v_1' = \frac{1}{t} - \frac{1}{3t^3}$$

$$v_1' = \frac{1}{t} - \frac{1}{3t^3}$$

$$v_2' = -t^2 + \frac{1}{3}$$

$$v_1 = \ln t + \frac{1}{6t^2} + C_1$$

$$v_2 = -\frac{t^3}{3} + \frac{t}{3} + C_2$$

$$\begin{aligned} y(t) &= v_1 y_1 + v_2 y_2 = t^2 \ln t + \frac{1}{6} + C_1 t^2 - \frac{t^2}{3} + \frac{1}{3} + C_2 t^{-1} \\ &= \underbrace{t^2 \left(\ln t - \frac{1}{3} \right) + \frac{1}{2}}_{y_{\text{particular}}} + \underbrace{C_1 t^2 + C_2 t^{-1}}_{y_{\text{hom}}} \end{aligned}$$

In each of Problems 29 through 32, use the method outlined in Problem 28 to solve the given differential equation.

3.6. 29. $t^2 y'' - 2ty' + 2y = 4t^2$, $t > 0$; $y_1(t) = t$

reduction of order

$$y(t) = v(t) y_1(t)$$

$$y = v y_1 = vt$$

$$y' = v't + v$$

$$y'' = v''t + v' + v'$$

$$t^2(v''t + \underline{2v'}) - 2t(\underline{v't} + \underline{v}) + 2(\underline{vt}) = 4t^2$$

$$v''t^3 = 4t^2$$

$$v'' = \frac{4}{t}$$

$$v' = 4 \ln t + C_1$$

$$v = 4(t \ln t - t) + C_1 t + C_2$$

$$y(t) = vt = 4(t^2 \ln t - t^2) + C_1 t^2 + C_2 t$$

Ex Var. Par. for 3rd order DE - how to choose constraints?

We have $y_{\text{hom}} = c_1 y_1 + c_2 y_2 + c_3 y_3$

3 unknowns

Assume $y(t) = u y_1 + v y_2 + w y_3$ where u, v, w are functions of x .

$$y'(t) = u y_1' + v y_2' + w y_3' + \underbrace{u' y_1 + v' y_2 + w' y_3}_{=0 \text{ (1st condition)}}$$

$$y''(t) = u y_1'' + v y_2'' + w y_3'' + \underbrace{u' y_1' + v' y_2' + w' y_3'}_{=0 \text{ (2nd cond.)}}$$

$$y'''(t) = u y_1''' + v y_2''' + w y_3''' + u' y_1'' + v' y_2'' + w' y_3''$$

this part gives 0 when we plug in

Ex. &
Uniq.
 n^{th} order
linear
DE

Theorem 4.1.1

If the functions $p_1, p_2, \dots, p_n$, and g are continuous on the open interval I , then there exists exactly one solution $y = \phi(t)$ of the differential equation (2) that also satisfies the initial conditions (3), where t_0 is any point in I . This solution exists throughout the interval I .

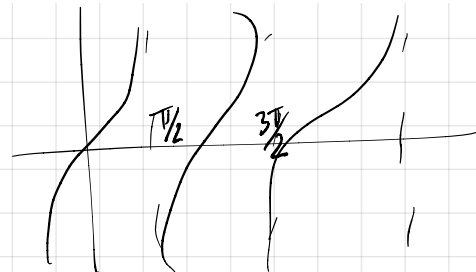
$$1. y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \dots + p_n y = g$$

In each of Problems 1 through 6, determine intervals in which solutions are sure to exist.

4.1.5. $(x-1)y^{(4)} + (x+1)y'' + (\tan x)y = 0$

4th order, lin. ODE

$$y^{(4)} + \underbrace{\frac{x+1}{x-1}}_{\text{discts if } x=1} y'' + \underbrace{\frac{\tan x}{x-1}}_{x=1 \text{ \& } x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}} y = 0$$



$$\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \text{ or } \left(-\frac{\pi}{2}, 1\right) \text{ or } \left(1, \frac{\pi}{2}\right) \text{ or } \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \dots$$

$$\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right) \text{ for } k \neq 0$$

$$\text{and } \left(-\frac{\pi}{2}, 1\right) \text{ and } \left(1, \frac{\pi}{2}\right)$$

In each of Problems 7 through 10, determine whether the given functions are linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

4.1.9. $f_1(t) = 2t - 3$, $f_2(t) = t^2 + 1$, $f_3(t) = 2t^2 - t$, $f_4(t) = t^2 + t + 1$ $\{1, t, t^2\}$

$$c_1 f_1 + c_2 f_2 + c_3 f_3 + c_4 f_4 = 0 \quad ? (c_1, c_2, c_3, c_4) \neq (0, 0, 0, 0)$$

$$c_1(2t-3) + c_2(t^2+1) + c_3(2t^2-t) + c_4(t^2+t+1) = 0$$

$\Rightarrow$ if all $c_i = 0$ then lin. independent.

if $\exists c_i \neq 0$ then lin. dependent.

if there is

$$\underbrace{t^2(c_2 + 2c_3 + c_4)}_{=0} + \underbrace{t(2c_1 - c_3 + c_4)}_{=0} + \underbrace{1(-3c_1 + c_2 + c_4)}_{=0} = 0t^2 + 0t + 0$$

4 unknowns c_1, c_2, c_3, c_4 & 3 constraints $\rightarrow$ degree of freedom ≥ 1

$$c_1 = -\frac{2}{3}, \quad c_2 = -\frac{13}{3}, \quad c_3 = 1, \quad c_4 = \frac{7}{3} \text{ solves it.}$$

In each of Problems 21 through 24, use Abel's formula (Problem 20) to find the Wronskian of a fundamental set of solutions of the given differential equation.

4.1.23. $\frac{1}{t}y''' + \frac{2}{t}y'' - \frac{1}{t}y' + \frac{1}{t}y = 0$

Problem 20 (d) Generalize this argument to the n th order equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_n(t)y = 0$$

with solutions $y_1, \dots, y_n$. That is, establish Abel's formula

$$W(y_1, \dots, y_n)(t) = c \exp \left[- \int p_1(t) dt \right]$$

for this case.

$$\begin{aligned} W(y_1, y_2, y_3)(t) &= c e^{-\int \frac{2}{t} dt} = c e^{-2 \ln|t| + c} \\ &= c t^{-2} \end{aligned}$$

In each of Problems 27 and 28, use the method of reduction of order (Problem 26) to solve the given differential equation.

4.1. 27. $(2-t)y''' + (2t-3)y'' - ty' + y = 0, \quad t < 2; \quad y_1(t) = e^t$

$$y = v y_1 = v e^t$$

$$y' = v' e^t + v e^t$$

$$y'' = v'' e^t + 2v' e^t + v e^t$$

$$y''' = v''' e^t + 3v'' e^t + 3v' e^t + v e^t$$

$$(2-t)(v''' e^t + 3v'' e^t + 3v' e^t + v e^t)$$

$$+ (2t-3)(v'' e^t + 2v' e^t + v e^t) - t(v' e^t + v e^t) + v e^t = 0$$

$$(2-t)(\underline{v'''} + \underline{3v''} + \underline{3v'} + v) + (2t-3)(\underline{v''} + \underline{2v'} + v) - t(\underline{v'} + v) + v = 0$$

$$(2-t)v''' + (3-t)v'' + (\cancel{6-3t+4t-6-t})v' + (\cancel{2-t+2t-3-t+1})v = 0$$

$$(2-t)v''' + (3-t)v'' = 0 \quad \longrightarrow \quad (2-t)u' + (3-t)u = 0 \quad u = v''$$

$$(2-t)v''' = (t-3)v''$$

$$v'' = u \quad \frac{u'}{u} = \frac{v'''}{v''} = \frac{t-3}{2-t} = -1 - \frac{1}{2-t}$$

$$\ln u = -t + \ln(2-t) + C \quad \text{ln|2-t|, and } t < 2$$

$$\int \frac{v''}{dt} = \quad u = e^{-t} \cdot \underbrace{e^{\ln(2-t)}}_{2-t} \cdot \underbrace{e^C}_C = C e^{-t} (2-t) \quad \int dt$$

$$v' = \int c_1 e^{-t} (2-t) dt$$

$$= c_1 \int 2e^{-t} - te^{-t} dt$$

$$= -2c_1 e^{-t} - c_1 (-te^{-t} - e^{-t} + c_2)$$

$$= -c_1 e^{-t} + c_1 t e^{-t} - c_2$$

$$v = \int -c_1 e^{-t} + c_1 t e^{-t} - c_2 dt$$

$$= \cancel{c_1 e^{-t}} + c_1 (-\cancel{te^{-t}} - \cancel{e^{-t}} + c_3) - c_2 t$$

$$= -c_1 t e^{-t} + c_1 - c_2^* t$$

$$y = v y_1 = (-c_1 t e^{-t} + c_2 - c_3 t) e^t$$

$$= \underbrace{-c_1 t}_{y_2} + \underbrace{c_2 e^t}_{y_1} - \underbrace{c_3 t e^t}_{y_3}$$

$$\int \overset{v}{t} \overset{du}{e^{-t}} dt = -te^{-t} + \int e^{-t} dt$$

$dv = dt$
 $u = -e^{-t}$

$$\overset{uv - \int u dv}{=} -te^{-t} - e^{-t} + c$$