Boğaziçi University
Department of Mathematics
Math 338 Complex Analysis
Spring 2024 – Final Exam

1	2	3	4	$\sum$
25 pts	25 pts	25 pts	25 pts	100 pts

Date: May 28, 2024 Time: 16:15-18:45	Full Name: PROPOSED SOLUTIONS

You may use every fact that we have already proven in the class.
In Questions 2,3,4 only the residue techniques are allowed to evaluate the integrals.

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1. Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $N > 0$ with distinct roots $z_1, \dots, z_N$; and $P'(z) = \frac{d}{dz}P(z)$ with a root α.

(a) For w not a root of P, show $\dfrac{P'(w)}{P(w)} = \sum_{k=1}^{N} (w - z_k)^{-1}$.

(b) Show that $\overline{\alpha} = \sum_{j=1}^{N} \beta_j \overline{z_j}$ where $\beta_j = \dfrac{|\alpha - z_j|^{-2}}{\sum_{k=1}^{N} |\alpha - z_k|^{-2}}$.

(c) Conclude that any root α of P' is a convex combination of the roots of P, i.e. α can be expressed as a linear combination of the roots of P with (i) each coefficient is real (ii) and nonnegative, (iii) and so that the sum of the coefficients is equal to 1. This is the Gauss-Lucas Theorem. Félix Lucas is a 19th century French mathematician.

(a) Write $P(z) = (z - z_1) \cdots (z - z_n)$. Then
$P'(\omega) = P(\omega) \cdot \sum_{k=1}^{n} 1/(\omega - z_k)$ for $\omega \neq z_k \ \forall k$.

(b) Let α be a root of $P'(z)$ which is not a root of $P(z)$.
Then $0 = P'(\alpha)/P(\alpha) = \sum (\alpha - z_k)^{-1} = \sum \overline{\alpha} - \overline{z}_k / |\alpha - z_k|^2$

$\Rightarrow \overline{\alpha} \cdot \sum_{k=1}^{n} \underbrace{\dfrac{1}{|\alpha - z_k|^2}}_{u_k} = \sum_{j=1}^{n} \dfrac{\overline{z}_j}{|\alpha - z_j|^2} \Rightarrow \alpha = \sum_{j=1}^{n} \dfrac{u_j}{\sum_k u_k} z_k$

(c) If α is also a root of $P(z)$, $\alpha = z_j$ & the claim is true.
Otherwise we just need to observe that in (b),

- $\beta_j = \dfrac{u_j}{\sum_k u_k}$ is in $[0,1]$. But this is obvious: since $\forall u_j$ is in $[0,1]$ and $0 \leqslant u_j \leqslant \sum_k u_k$
- $\sum \beta_j = 1$ obvious.

Watch this super joyful video about loci of roots of derivatives:
https://en.wikipedia.org/wiki/File:Gauss_Lucas_Theorem_animation.webm

2. (a) Determine the **poles** of $f(z) = \dfrac{1}{z^4+z^2+1}$. **Plot** them on $\mathbb{C}$. Evaluate the **residue** of f at <u>each</u> pole.

(b) Evaluate the real integral $\displaystyle\int_0^\infty \frac{dx}{x^4+x^2+1}$. For this choose an appropriate contour in $\mathbb{C}$, and evaluate the corresponding integral by carefully arguing why some portion of the integral vanishes.

(Answer: $\sqrt{3}\pi/6$)

(a) $z^4+z^2+1=0 \iff z^2 = \frac{1}{2}(-1\pm\sqrt{1-4}) = \frac{1}{2}(-1\pm i\sqrt{3}) = e^{\pm i 2\pi/3}$

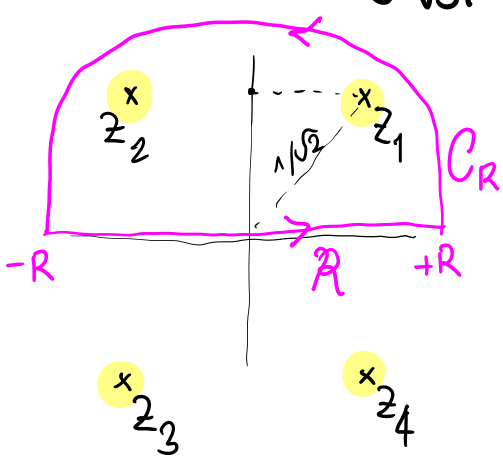
$\iff z = \begin{cases} e^{\pm i\cdot\pi/3} \\ e^{\pm i\cdot 4\pi/3} \end{cases} = \pm 1 \pm \sqrt{3}i$: Four distinct simple poles

For ω a pole, $\mathrm{res}(f;\omega) = \dfrac{1}{4\omega^3+2\omega} = \dfrac{\omega}{4\omega^4+2\omega^2} = \dfrac{-\omega}{2\omega^2+4}$ $\quad 3\pm i\sqrt{3}$ $\quad -1\pm i\sqrt{3}$

So $\mathrm{res}(f;z_1) = -\dfrac{(1+\sqrt{3}i)/2}{3+\sqrt{3}i} = \dfrac{-1}{12}\cdot(3+\sqrt{3}i)$ $\quad \dots$

$\mathrm{res}(f;z_2) = -\dfrac{(-1+\sqrt{3}i)/2}{3-\sqrt{3}i} = \dfrac{1}{12}(3-\sqrt{3}i)$

(b)



Consider the contour $C = \mathbb{R} \cup C_R$ oriented positively to evaluate $\int_{-\infty}^{+\infty} f(x)dx$ & divide the result by 2 since f is an even function.

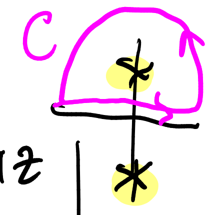
- $\oint_C f\,dz = 2\pi i\,(\mathrm{res}(f;z_1) + \mathrm{res}(f;z_2))$
  $= \dfrac{2\pi i}{12}\cdot(-2\sqrt{3}i) = \pi/\sqrt{3}$

- $\left|\int_{C_R} f\,dz\right| \leq \pi.R.\dfrac{1}{R^4-R^2-1} \xrightarrow{R\to\infty} 0$.
  R large

Hence $\displaystyle\int_0^\infty f(x)\,dx = \frac{1}{2}\int_C f\,dz = \pi/2\sqrt{3}$

3. (a) Determine the **poles** of $g(z) = \frac{e^{iaz}}{(z^2+1)^2}$. **Plot** them on $\mathbb{C}$. Evaluate the **residue** of g at each pole.

(b) Evaluate the real integral $\int_0^\infty \frac{\cos ax}{(x^2+1)^2}dx$ with $a \in \mathbb{R}^{\geq 0}$. For this choose an appropriate contour in $\mathbb{C}$, and evaluate a corresponding integral by carefully arguing why some portion of the integral vanishes. How did you use $a > 0$? (Answer: $\frac{\pi(a+1)}{4e^a}$)



(a) Two poles of order 2: $z_\pm = \pm i$,

$$\mathrm{res}(g;\pm i) = \frac{d}{dz}\left[(z\mp i)^2 \cdot g(z)\right]_{z=\pm i} = \frac{iae^{iaz}}{(z\pm i)^2} - \frac{2e^{iaz}}{(z\pm i)^3}\Bigg|_{z=\pm i}$$

$$= \frac{iae^{\mp a}}{-4} - \frac{e^{\mp a}}{\mp 4i} = \frac{-e^{\mp a}}{4i}\cdot\left[a \pm 1\right]$$

(b) Since g is even, first we evaluate $\int_C \frac{e^{iaz}}{(z^2+1)^2}dz$,

- $$\int_C \frac{e^{iaz}}{(z^2+1)^2}dz = 2\pi i \cdot \mathrm{res}(g;+i) = \pi \cdot e^{-a} \cdot \frac{a+1}{2}$$
- $$\left|\int_{C_R} \frac{e^{iaz}}{(z^2+1)^2}dz\right| \leq \pi \cdot R \cdot \frac{|e^{-ay}|}{(R^2-1)^2} \leq \pi R \cdot \frac{1}{(R^2-1)^2} \xrightarrow{R\to\infty} 0$$

$$|e^{iaz}| = |e^{iax}\cdot e^{-ay}| = |e^{-ay}| \leq 1 \quad \forall y>0,$$
when $a>0$.

Now
$$\mathrm{Re}\int_C \frac{e^{iaz}}{(z^2+1)}dz = \mathrm{Re}\int_{-\infty}^{+\infty} \frac{e^{iax}}{(x^2+1)}dx = \int_{-\infty}^{+\infty} \frac{\cos ax}{(x^2+1)^2}dx$$

Hence
$$\int_0^\infty \frac{\cos ax}{(x^2+1)^2}dx = \pi(a+1)/4e^a$$

4. Let $\mathcal{C}$ be the contour $z = e^{i\theta}, \theta \in [0, 2\pi]$, i.e. $|z| = 1$.

(a) Show that on $\mathcal{C}$, $\cos\theta = (z + 1/z)/2$.

(b) For b a constant, find two polynomials $U(z), V(z)$ such that

$$I = \int_0^{2\pi} \frac{d\theta}{(b+\cos\theta)^2} = \int_{\mathcal{C}} \frac{U(z)}{V(z)} dz.$$

(Hint: Do not forget the Jacobian.)

(c) For $b \in \mathbb{R}$, $\overset{b>1}{\cancel{|b|>}}$ 1, evaluate the real integral $\int_0^{2\pi} \frac{dx}{(b+\cos x)^2}$. How did you use the condition on b?

(Answer: $\frac{2\pi b}{(b^2-1)^{3/2}}$)

(a) $\cos\theta = (e^{i\theta} + e^{-i\theta})/2 = (z + \frac{1}{z})/2$

(b) On C, $\frac{1}{(b+\cos z)^2} = \frac{4}{(2b+z+\frac{1}{z})^2} = \frac{4z^2}{(z^2+2bz+1)^2}$

Now observe:

$$\int_{|z|=1} \frac{4z/i}{(z^2+2bz+1)^2} dz \underset{z=e^{i\theta}}{=} \int_0^{2\pi} \frac{4z/i}{(z^2+2bz+1)^2} \cdot \overbrace{ie^{i\theta}}^{iz} d\theta = \int_0^{2\pi} \frac{4z^2}{(z^2+2bz+1)^2} d\theta$$

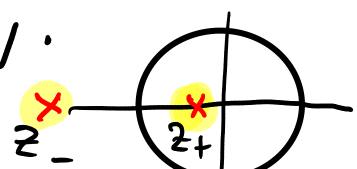
So if we set $U(z) = 4z/i$ & $V(z) = (z^2+2bz+1)^2$,

$\int_C U/V = I$.

(c) Following part (b), compute $\int_C U/V$.

$b^2-1>0$

$V=0 \iff z_\pm = -b \pm \sqrt{b^2-1} \in \mathbb{R}$



Two poles of order 2.

$$\operatorname{res}(U/V; z_+) = \frac{d}{dz}\left[(z-z_+)^2 \cdot U/V\right]_{z=z_+} = 4i\left[\frac{z}{(z-z_-)^2}\right]_{z=z_+}$$

$$= 4i\left[\frac{1}{(z_+-z_-)^2} - \frac{2z_+}{(z_+-z_-)^3}\right] = -4i \cdot \frac{z_+ + z_-}{(z_+-z_-)^3} = -4i \cdot \frac{-2b}{8(b^2-1)^{3/2}} = \frac{ib}{(b^2-1)^{3/2}}$$

Hence $I = 2\pi i \cdot \operatorname{res}(U/V; z_+) = \frac{-2\pi b}{(b^2-1)^{3/2}}$