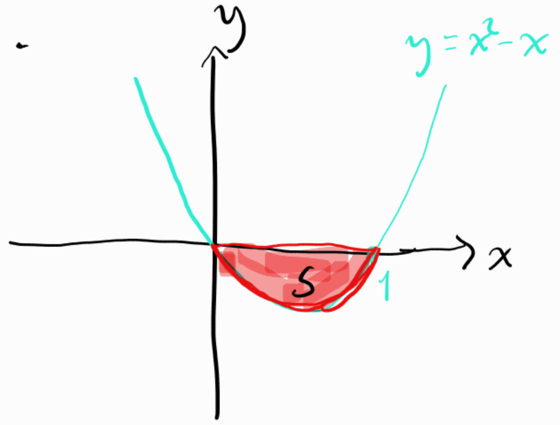
1. For each of the following sets S in the plane $\mathbb{R}^2$, do the following: (i) Draw a sketch of S. (ii) Tell whether S is open, closed, or neither. (iii) Describe S^{int}, $\overline{S}$, and ∂S. (These descriptions should be in the same set-theoretic language as the description of S itself given here.)

a. $S = \{(x, y) : 0 < x^2 + y^2 \leq 4\}$.
b. $S = \{(x, y) : x^2 - x \leq y \leq 0\}$.
c. $S = \{(x, y) : x > 0,\ y > 0, \text{ and } x + y > 1\}$.
d. $S = \{(x, y) : y = x^3\}$.
e. $S = \{(x, y) : x > 0 \text{ and } y = \sin(1/x)\}$.
f. $S = \{(x, y) : x^2 + y^2 < 1\} \setminus \{(x, 0) : x < 0\}$.
g. $S = \{(x, y) : x \text{ and } y \text{ are rational numbers in } [0, 1]\}$.

b.

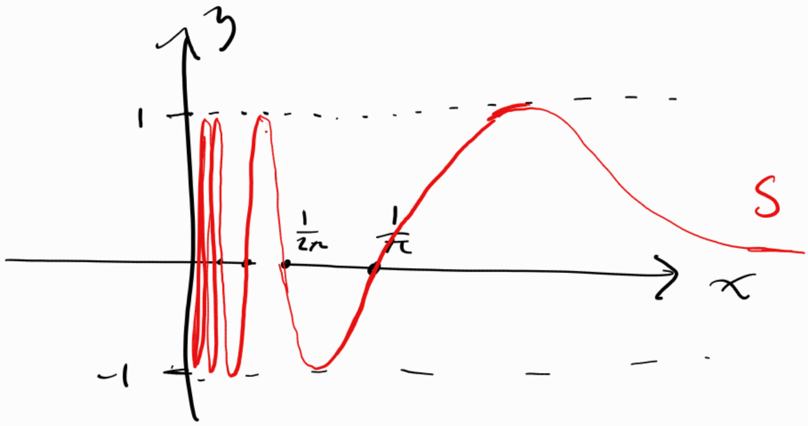


S is closed.

$$\partial S = \left\{ (x,y) \;\middle|\; \begin{array}{c} x^2 - x = y < 0 \\ \text{or} \\ x^2 - x \leq y = 0 \end{array} \right\}$$

$$S^{\text{int}} = S \setminus \partial S$$
$$= \{(x,y) : x^2 - x < y < 0\}$$

e.

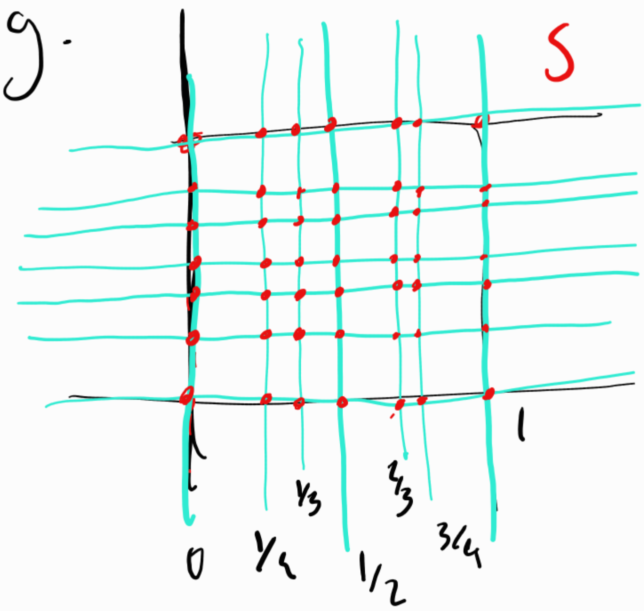


S is neither closed nor open.

$$\partial S = S \cup \{(0,y) : y \in [-1,1]\}$$

$$S^{\text{int}} = \emptyset \qquad \overline{S} = S \cup \partial S.$$

9.



S is neither closed nor open.

$$\partial S = [0,1] \times [0,1]$$
$$= \{(x,y) : 0 \leq x, y \leq 1\}.$$

$$S^{int} = \emptyset.$$

$$\overline{S} = \partial S.$$

<u>Proof that $\partial S = [0,1] \times [0,1]$:</u>

Take $p = (x, y) \in [0,1] \times [0,1]$.

Let $r > 0$. Observe that

$$B(\tfrac{r}{2}, x) \times B(\tfrac{r}{2}, y) \subset B(r, p).$$

It suffices to show that for each $x \in \mathbb{R}$, and $\varepsilon > 0$, there is some $q \in B(\varepsilon, x) \cap \mathbb{Q}$ and $\eta \in B(\varepsilon, x) \cap \mathbb{Q}^c$.

Let $N \in \mathbb{N}$ be large enough so that $10^{-N} < \varepsilon$.

Consider the decimal expansion $x = \pm \sum_{n=-k}^{\infty} x_n 10^{-n}$.

$(x_n \in \{0, 1, \ldots, 9\})$

Clearly $q = \pm \sum_{n=-k}^{N} x_n 10^{-n}$ satisfies

$$|x-q| = \sum_{n=N+1}^{\infty} x_n 10^{-n} \leq 10^{-N} < \varepsilon ,$$

Hence $q \in B(\varepsilon, x) \cap \mathbb{Q}$.

Also we have $B(\varepsilon, x) \cap \mathbb{Q}^c$ as

$\mathbb{Q}$ is countably infinite while

$B(\varepsilon, x)$ is not.

2. Show that for any $S \subset \mathbb{R}^n$, S^{int} is open and ∂S and $\overline{S}$ are both closed. (*Hint:* Use the fact that balls are open, proved in Example 1.)

1.4 Proposition. *Suppose $S \subset \mathbb{R}^n$.*

a. S is open $\iff$ every point of S is an interior point.

b. S is closed $\iff$ S^c is open.

$\underline{S^{int}}$: Let $x \in S^{int}$. By ,

it suffices to show that

$x \in (S^{int})^{int}$. As $x \in S^{int}$,

there is some $r > 0$ s.t the ball $B(r, x) \subset S$.

As $B(r, x)$ is open, each point

$y \in B(r, x)$ is interior. Hence

$y \in S^{int}$ also. Hence $B(r, x) \subset S^{int}$.

Hence $x \in (S^{int})^{int}$.

$\overline{S}$ is closed.

Let $x \in \partial(\overline{S})$. Suppose that $x \notin \overline{S}$ for a contradiction. Then $x \notin S$ and $x \notin \partial S$. It follows that there is some $r > 0$ s.t $\boxed{B(r,x) \subset S^c}$.

Pick $a \in B(r,x) \cap \overline{S} \neq \emptyset$.

Since $B(r,x) \subset S^c$, we must have $a \in B(r,x) \cap \partial S$.

Let $r' = r - |a-x|$. Observe that $B(r',a) \subset B(r,x)$. But since $a \in \partial S$, $B(r',a) \cap S \neq \emptyset$. Contradicting

$\boxed{}$.

∂S is closed.

Observe that $\partial S = \overline{S} \setminus S^{int}$.

Indeed, if $x \in \partial S$, then $x \notin S^{int}$;

and if $x \in \overline{S} \setminus \partial S$, then

there must be some $r > 0$ s.t

$B(r,x) \cap S^c = \emptyset$, that is, $B(r,x) \subset S$,

so $x \in S^{int}$.

So $\partial S = \overline{S} \cap (S^{int})^c$

Recall that $\overline{S}$ is closed.

And S^{int} is open, hence $(S^{int})^c$

is closed. We proved in class that

the intersection of closed sets is closed.

Hence ∂S is closed.

Bonus: Let $S \subset \mathbb{R}^n$. The closure $\overline{S}$ is the smallest closed set containing S. That is, if $T \supset S$ is closed, then $T \supset \overline{S}$.

5. Show that the boundary of S is the intersection of the closures of S and S^c.

6. Give an example of an infinite collection $S_1, S_2, \ldots$ of closed sets whose union $\bigcup_{j=1}^{\infty} S_j$ is not closed.

5. Observe that $\partial S = \partial S^c$.

Indeed, $x \in \partial S \overset{\text{(definition)}}{\iff} \forall r>0 \ B(r,x)\cap S \neq \emptyset$ and $B(r,x)\cap S^c \neq \emptyset$.

$\iff x \in \partial S^c$,

since $(S^c)^c = S$.

Follland defines $\overline{S} = S \cup \partial S$.

So $\overline{S^c} \cap \overline{S} = (S^c \cup \underbrace{\partial S^c}_{= \partial S}) \cap (S \cup \partial S)$

$= \underbrace{(S^c \cap S)}_{= \emptyset} \cup \underbrace{(S^c \cap \partial S)}_{\subset \partial S} \cup \underbrace{(\partial S \cap S)}_{\subset \partial S} \cup \underbrace{(\partial S \cap \partial S)}_{= \partial S}$

$= \partial S$.

6. Let $S_j = \{\frac{1}{j}\} \subset \mathbb{R}$.

Points are closed.

Call the union 'S'.

Claim: $0 \in \partial\left(\bigcup_{j=1}^{\infty} S_j\right)$.

Pf: Let $r > 0$. Consider the ball

$$B(r, 0) = (-r, r).$$

Clearly $B(r, 0) \cap S^c \ni 0$.

And, the is some $N \in \mathbb{N}$ s.t

$$\frac{1}{N} < r,$$ hence $\frac{1}{N} \in B(r,0) \cap S$.

Thus 0, as claimed, is in the bdry.

8. Give an example of a set S such that the interior of S is unequal to the interior of the closure of S.

Let $S = \{x \in \mathbb{R}^n : |x| \neq 1\}$.

Then $\operatorname{int} S = S \neq \mathbb{R}^n = \overline{S} = \operatorname{int} \overline{S}$.

Alternatively, consider $S = \mathbb{Q} \subset \mathbb{R}$.

$\operatorname{int} S = \emptyset$. But $\operatorname{int} \overline{S} = \overline{S} = \mathbb{R}$.