1. For the following functions f, show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

a. $f(x,y) = \dfrac{x^2+y}{\sqrt{x^2+y^2}}$

b. $f(x,y) = \dfrac{x}{x^4+y^4}$

c. $f(x,y) = \dfrac{x^4y^4}{(x^2+y^4)^3}$

2. For the following functions f, show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

a. $f(x,y) = \dfrac{x^2y^2}{x^2+y^2}$ b. $f(x,y) = \dfrac{3x^5 - xy^4}{x^4+y^4}$

1.c. Consider the curves $x = \alpha y^2$.

Taking the limits along these curves, we get

$$\lim_{(x=\alpha y^2,\, y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(\alpha y^2, y)$$

$$= \lim_{y\to 0} \frac{\alpha^4 y^{12}}{(\alpha+1)^3 y^{12}}$$

$$= \frac{\alpha^4}{(\alpha+1)^3},$$

The limit is different for $\alpha = 0$, and $\alpha = 1$.

Hence the limit does not exist.

2. a. Observe that $x^2, y^2 \geq 0$.

Hence $x^2 + y^2 \geq x^2 \geq 0$.

Consequently $0 \leq \dfrac{x^2}{x^2+y^2} \leq 1$ for all $(x,y) \in \mathbb{R}^2$.

Then $$0 \leq \frac{x^2 y^2}{x^2+y^2} \leq y^2.$$

So $0 \leq \lim_{(x,y)\to(0,0)} f(x,y) \leq \lim_{(x,y)\to(0,0)} y^2 = 0$.

Thus $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

3. Let $f(x,y) = x^{-1}\sin(xy)$ for $x \neq 0$. How should you define $f(0,y)$ for $y \in \mathbb{R}$ so as to make f a continuous function on all of $\mathbb{R}^2$?

Recall that $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

Furthermore, $\lim_{(x,y)\to(0,y_0)} xy = 0$. ☆

Hence $\lim_{(x,y)\to(0,y_0)} \frac{1}{y}\left(f(x,y) - y\right) = \lim_{(x,y)\to(0,y_0)} \frac{\sin(xy)}{xy} - 1$

$= \lim_{\substack{t\to 0 \\ \shortparallel \\ xy}} \frac{\sin t}{t} - 1 = 0.$

Follows from ☆ and composition of limits

Since $\lim_{(x,y)\to(0,y_0)} \frac{1}{y} \neq 0$,

We must have $\lim_{(x,y)\to(0,y_0)} f(x,y) - y = 0$.

As $\lim_{(x,y)\to(0,y_0)} y = y_0$,

we must have

$$\lim_{(x,y)\to(0,y_0)} f(x,y) = y_0.$$

Setting $f(0,y) = y$ is the only choice.

Clearly the extended function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

this we showed earlier

We have

$$\lim_{(x,y)\to(0,y_0)} f(x,y) = \lim_{\substack{(x,y)\to(0,y_0)\\(x\neq 0)}} f(x,y) = \lim_{\substack{(x,y)\to(0,y_0)\\x=0}} f(x,y)$$

if both limits exist and agree

this is too easy.

$$= y_0.$$

Bonus: Is there a contns function $f: \mathbb{R}^2 \setminus \{x=0\} \to \mathbb{R}$

s.t for each $y_0 \in \mathbb{R}$, the

limit $\lim_{(x,y)\to(0,y_0)} f(x,y) = g(y_0)$

exists, but the extension

$\hat{f}: \mathbb{R}^2 \to \mathbb{R}$

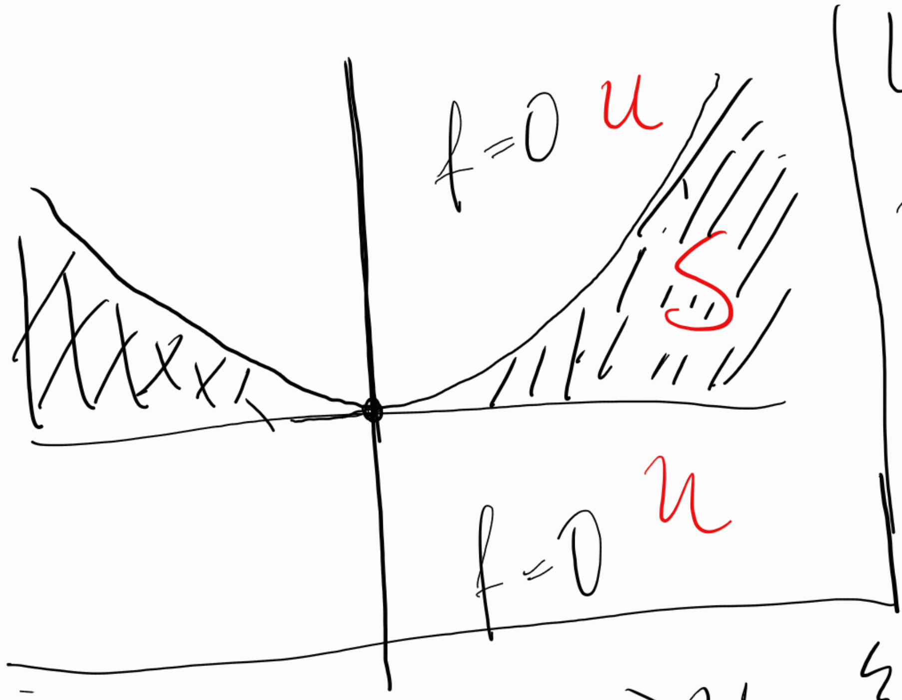
$$\hat{f}(x,y) = \begin{cases} f(x,y) & \text{if } x \neq 0 \\ g(y) & \text{if } x = 0 \end{cases}$$

is not continuous?

continuous in x and y.

5. Let $f(x,y) = y(y-x^2)/x^4$ if $0 < y < x^2$, $f(x,y) = 0$ otherwise. At which point(s) is f discontinuous?

6. Let $f(x) = x$ if x is rational, $f(x) = 0$ if x is irrational. Show that f



Let $S = \{0 < y < x^2\}$

$U = (S^c)^{int}$

$= \{y > x^2\} \cup \{y < 0\}$.

Then $\partial S = \partial U = \{y = 0\} \cup \{y = x^2\}$.

Observe that both S & U are open, and that f is continuous on both sets. So we investigate ∂S. Let $(a,b) \in \partial S$.

Then $\lim\limits_{\substack{(x,y)\to(a,b)\\(x,y)\in U}} f(x,y) = f(a,b) = 0$. So we must investigate $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ over $(x,y)\in S$.

Notice that the function $g(x,y) = y(y-x^2)$ satisfies

$\lim\limits_{(x,y)\to(a,b)} g(x,y) = 0$ whenever $(a,b)\in\partial S$. So as long as

$\frac{1}{x^4}$ remains bounded,

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} \frac{1}{x^4} g(x,y) = 0.$$

This is the case whenever $a \neq 0$.

For $a = 0$, as $(a,b) \in \partial S$, $b = 0$.

Let us take limits over

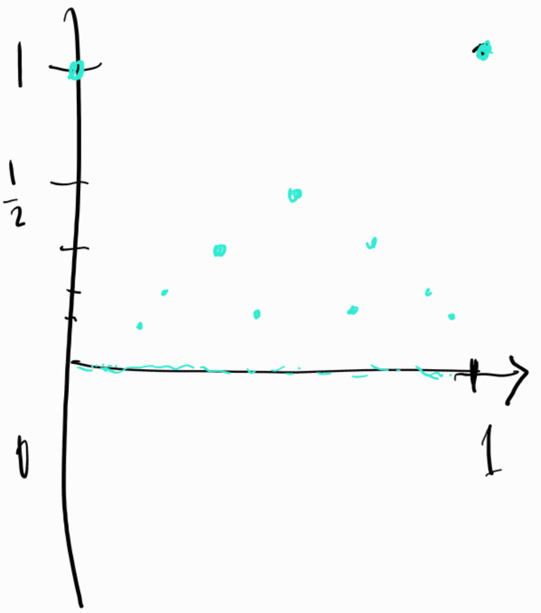
the curves $y = \alpha x^2$ for $\alpha \in (0,1)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, \alpha x^2)$$

$$= \lim_{x\to 0} \frac{\alpha(\alpha-1)x^4}{x^4} = \alpha(\alpha-1).$$

So the point $(0,0)$ is the only pt where f is discnts.

7. Let $f(x) = 1/q$ if $x = p/q$ where p and q are integers with no common factors and $q > 0$, and $f(x) = 0$ if x is irrational. At which points, if any, is f continuous?



i) f is discnts at all rationals.

Pf: Say f was conts at $x = p/q$. Then $f(x) = \frac{1}{q} > 0$.

By continuity, we must have $f(y) > 0$ for some interval $y \in (x-\delta, x+\delta)$, where $\delta > 0$. But this is not the case, as there is an irrational $z \in (x-\delta, x+\delta)$.

ii) f is contns at all irrationals.

Pf: Let $x \in \mathbb{R}$ be irrational, and $\varepsilon > 0$. Then is some integer $N > 0$ s.t. $\frac{1}{N} < \varepsilon$.

Observe that $0 \leq f(y) < \frac{1}{N}$ if $y \in \mathbb{R} \setminus \frac{1}{N!}\mathbb{Z}$, where $\frac{1}{N!}\mathbb{Z} = \{\frac{a}{N!} : a \in \mathbb{Z}\}$, and $N! = 1 \cdot 2 \cdot 3 \cdots N$.

Indeed, if y is irrational $f(y) = 0$.
And if $y = p/q$, then we must have $q > N$. Since if not, then $q \mid N!$ and it follows that $p/q = \frac{a}{N!}$ where $a = p \cdot (N!/q)$.
Hence $f(y) < \frac{1}{N}$ as stated.

Observe that $\mathbb{R} \setminus \frac{1}{N!}\mathbb{Z}$ is open, hence there is some interval $(x - \delta, x + \delta) \subset \mathbb{R} \setminus \frac{1}{N!}\mathbb{Z}$. $\blacksquare$

continuous?

8. Suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^k$ has the following property: For any open set $U \subset \mathbb{R}^k$, $\{\mathbf{x} : \mathbf{f}(\mathbf{x}) \in U\}$ is an open set in $\mathbb{R}^n$. Show that $\mathbf{f}$ is continuous on $\mathbb{R}^n$. Show also that the same result holds if "open" is replaced by "closed."

Let $x \in \mathbb{R}^n$, $\varepsilon > 0$.

Let $V = \{ x \in \mathbb{R}^n : f(x) \in B(f(x), \varepsilon)\}$

V is open, and $x \in V$. Hence there is some $\delta > 0$ s.t. $B(x, \delta) \subset V$.

So f is cnts.

– Replace 'open' w/ 'closed')

(Recall the notation $f^{-1}(S)$, where $f : A \to B$ is a function, $S \subset B$ and $f^{-1}(S) = \{a \in A : f(a) \in S\}$).

Let $\mathcal{U} \subset \mathbb{R}^k$ be open. Then $\mathcal{U}^c$ is closed. Hence so is $f^{-1}(\mathcal{U}^c) = f^{-1}(\mathcal{U})^c$.

So $f^{-1}(\mathcal{U})$ is open. We arrived at the original hypothesis.

Recall:

1.13 Theorem. *Suppose* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^k$ *is continuous and* U *is a subset of* $\mathbb{R}^k$*, and let* $S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{f}(\mathbf{x}) \in U\}$*. Then* S *is open if* U *is open, and* S *is closed if* U *is closed.*

So we have a different characterization of continuity.

9. Let U and V be open sets in $\mathbb{R}^n$ and let $\mathbf{f}$ be a one-to-one mapping from U onto V (so that there is an inverse mapping $\mathbf{f}^{-1} : V \to U$). Suppose that $\mathbf{f}$ and $\mathbf{f}^{-1}$ are both continuous. Show that for any set S such that $\overline{S} \subset U$ and $\overline{\mathbf{f}(S)} \subset V$ we have $\mathbf{f}(\partial S) = \partial(\mathbf{f}(S))$.

<u>Remark</u>. The assumption $\overline{f(S)} \subset V$ is unnecessary. It follows from the fact that $f(\overline{S}) = (f^{-1})^{-1}(\overline{S})$ is closed and contains $f(S)$. Hence $f(\overline{S}) \supset \overline{f(S)}$.

Since f^{-1} is continuous, for an open $A \subset U$ and closed $K \subset U$, $f(A) = (f^{-1})^{-1}(A)$ is open, and $f(K) = (f^{-1})^{-1}(K)$ is closed.

$f^{-1}(\overline{f(S)}) \subset U$ is closed,

and contains S. Hence

$$f^{-1}(\overline{f(S)}) \supset \overline{S}.$$

But also $f(\overline{S}) \supset \overline{f(S)}$.

Hence $f(\overline{S}) = \overline{f(S)}$.

Since $f(S^{int}) \subset f(S)$ is

open, $f(S^{int}) \subset f(S)^{int}$.

Also $f^{-1}(f(S)^{int}) \subset S$ is open.

Hence $f^{-1}(f(S)^{int}) \subset S^{int}$.

Hence $f(S^{int}) = f(S)^{int}$.

Then

$$\partial f(S) = \overline{f(S)} \setminus f(S)^{int}$$

$$= f(\overline{S}) \setminus f(S^{int})$$

$$= f(\overline{S} \setminus S^{int})$$

$$= f(\partial S).$$