4. Let $\{x_k\}$ and $\{y_k\}$ be sequences in $\mathbb{R}$ such that $x_k \to a$ and $y_k \to b$. Show that $x_k + y_k \to a + b$ and $x_k y_k \to ab$. (Use Theorems 1.10 and 1.15.)

1.10 Theorem. *Let $f_1(x, y) = x + y$, $f_2(x, y) = xy$, and $g(x) = 1/x$. Then f_1 and f_2 are continuous on $\mathbb{R}^2$ and g is continuous on $\mathbb{R} \setminus \{0\}$.*

1.15 Theorem. *Given $S \subset \mathbb{R}^n$, $\mathbf{a} \in S$, and $\mathbf{f} : S \to \mathbb{R}^m$, the following are equivalent:*

a. $\mathbf{f}$ *is continuous at* $\mathbf{a}$.

b. For any sequence $\{\mathbf{x}_k\}$ in S that converges to $\mathbf{a}$, the sequence $\{\mathbf{f}(\mathbf{x}_k)\}$ converges to $\mathbf{f}(\mathbf{a})$.

Apply 1.15 to the case

$S = \mathbb{R}^2$, $f = f_1$ or $f = f_2$

and $\mathbf{a} = (a, b)$.

A point $\mathbf{a} \in \mathbb{R}^n$ is called an **accumulation point** of a set $S \subset \mathbb{R}^n$ if every neighborhood of $\mathbf{a}$ contains infinitely many points of S. (The point $\mathbf{a}$ itself may or may not belong to S. Some people use the terms "limit point" or "cluster point" instead of "accumulation point.") For example, the accumulation points of the interval $(-1, 1)$ in $\mathbb{R}$ are the points in the closed interval $[-1, 1]$, and the only accumulation point of the set $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ is 0.

6. Show that $\mathbf{a}$ is an accumulation point of S if and only if there is a sequence $\{\mathbf{x}_k\}$ of points in S, none of which are equal to $\mathbf{a}$, such that $\mathbf{x}_k \to \mathbf{a}$. (Adapt the proof of Theorem 1.14.)

7. Show that the closure of S is the union of S and the set of all its accumulation points.

Equivalent definition: $x \in \mathbb{R}^n$ is an accumulation pt if every nbhood of x contains a point of S other than x

6. Let a be an accumulation point of S. Let $x_k \in B(\frac{1}{k}, a) \cap S \setminus \{a\}$.

Then $\lim_{k \to \infty} x_k = a$.

Let $x_k \in S \setminus \{a\}$ be a sequence of points s.t. $\lim_{k \to \infty} x_k = a$. Let N be a nbhood of a.

Then there is some $\varepsilon > 0$ so that

$B(\varepsilon, a) \subset N$. As $\lim_{k\to\infty} x_k = a$,

there is some K s.t for $k \geq K$

$x_k \in B(\varepsilon, a)$.

Clearly the set $\{x_k : k \geq K\}$ is an infinite collection. If not, the sequence would have to repeat a certain value, say x_{k_0}, infinitely often. But this contradicts the fact that $\lim_{k\to\infty} x_k = a$, as $x_{k_0} \neq a$.

7. Recall the characterization

1.14 Theorem. *Suppose $S \subset \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. Then $\mathbf{x}$ belongs to the closure of S if and only if there is a sequence of points in S that converges to $\mathbf{x}$.*

By the above,

$a \in \overline{S} \iff \exists$ a sequence $x_k \in S$ $\lim_{k\to\infty} x_k = a$.

$\iff a \in S$ or $\exists$ a sequence $x_k \in S$ $\lim_{k\to\infty} x_k = a$.

$\Longleftrightarrow a \in S$ or ($a \notin S$ and $\exists$ a sequence $x_k \in S$ $\lim_{k\to\infty} x_k = a$)

$\Longleftrightarrow a \in S$ or a is an accumulation pt of S (by 6).

$a \in S \Rightarrow \exists$ a sequence $x_k \in S$ $\lim_{k\to\infty} x_k = a$.
(take $x_k = a$)

For any propositions p, q we have
$p \vee q \Longleftrightarrow p \vee (\neg p \wedge q)$.

The following are from

Pólya, Szegö's book

"Problems and Theorems in Analysis (Vol I)".

67. The existence of $\lim_{n\to\infty} s_n$ implies

$$\lim_{n\to\infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} = \lim_{n\to\infty} s_n .$$

68. If the sequence $p_1, p_2, \ldots, p_n, \ldots$ of positive numbers converges to the positive value p then

$$\lim_{n\to\infty} \sqrt[n+1]{p_0 p_1 p_2 \cdots p_n} = p .$$

68.1. The numbers $a_0, a_1, a_2, \ldots$ are positive and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = p .$$

Then $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists also and has the same value p.

69. Reduce the computation of $\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}}$ to the computation of $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

67. Let $\varepsilon > 0$. Denote $s := \lim_{n\to\infty} s$.

Let N be large enough so that

$|s - s_n| < \varepsilon$ for $n \geq N$.

Observe now that for $n \geq N$

$$\left| \frac{s_0 + s_1 + \cdots + s_n}{n+1} - s \right|$$

$$= \left| \frac{s_0 + \cdots + s_N}{n+1} - \frac{(N+1)s}{n+1} + \frac{s_{N+1} - s + \cdots + s_n - s}{n+1} \right|$$

$$\leq \frac{|s_0 + \cdots + s_N|}{n+1} - \frac{N}{n+1}|s| + \frac{\varepsilon(n-N)}{n+1},$$

As $n \to \infty$, we have

$$\frac{|s_0 + \cdots + s_N|}{n+1} - \frac{N}{n+1}|s| \to 0,$$

since N is fixed. Therefore one can find some M s.t for $n \geq M$, $\frac{|s_0 + \cdots + s_N|}{n+1} - \frac{N}{n+1}|s| < \varepsilon$.

Assuming $n \geq \max\{M, N\}$, we get

$$\left| \frac{s_0 + s_1 + \cdots + s_n}{n+1} - s \right| < 2\varepsilon.$$

$\blacksquare$

68. Let $s_n = \log p_n$. Apply 67.

68.1 Observe that $a_n = a_0 \left(\frac{a_1}{a_0} \cdot \frac{a_2}{a_1} \cdots \frac{a_n}{a_{n-1}} \right)$

Denote $p_i = \frac{p_{i+1}}{p_i}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \underbrace{\lim_{n\to\infty} \sqrt[n]{a_0}}_{=1} \underbrace{\lim_{n\to\infty} \sqrt[n]{p_0 \cdots p_{n-1}}}_{\overset{\text{using } 68}{=} p}$$

$$= p.$$

69. Let $a_n = \frac{n^n}{n!}$.

Then $\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!}$

$$= (1 + 1/n)^n.$$

Apply 68.1.