QUIZ#2 — a solution

$$f(z)=\begin{cases} \bar{z}^2/z = \dfrac{1}{|z|^2}\bar{z}^3 = \dfrac{1}{x^2+y^2}(x-iy)^3 & \\ 0\ , & z=0 \end{cases}$$

$\Delta = x^2+y^2$

$$= \frac{1}{\Delta}\Big[\underbrace{(x^3+3xy^2)}_{u\cdot\Delta} + i\underbrace{(-3x^2y+y^3)}_{v\cdot\Delta}\Big]$$

(a) Away from $(0,0)$:

$$u_x = \frac{1}{\Delta}(3x^2+3y^2) - \frac{2x}{\Delta^2}\cdot(x^3+3xy^2)$$
$$= \frac{1}{\Delta}\left[3x^4+3y^4+6x^2y^2-2x^4-6x^2y^2\right]$$
$$= \frac{1}{\Delta}\cdot(x^4+3y^4)$$

$$v_y = \frac{1}{\Delta}(3y^2-3x^2) - \frac{2y}{\Delta^2}(-3x^2y+y^3)$$
$$= \frac{1}{\Delta}\left[-3x^4+3y^4+6x^2y^2-2y^4\right]$$
$$= \frac{1}{\Delta}\cdot(y^4+6x^2y^2-3x^4)$$

CR1: $u_x = v_y$

$$2x^4-3x^2y^2+y^4=0$$
$$\Leftrightarrow (2x^2-y^2)(x^2-y^2)=0$$
$$\Leftrightarrow \boxed{x^2=2y^2} \text{ OR } \boxed{x^2=y^2}$$

$$u_y = \frac{1}{\Delta}6xy - \frac{2y}{\Delta^2}(x^3+3xy^2)$$
$$= \frac{1}{\Delta}(6x^3y+\cancel{6xy^3}-2yx^3-\cancel{6xy^3}) = \frac{4yx^3}{\Delta}$$

$$v_x = \frac{1}{\Delta}(-6xy) - \frac{2x}{\Delta^2}(-3x^2y+y^3)$$
$$= \frac{1}{\Delta}(-6x^3y-6xy^3+6x^3y-2xy^3) = \frac{-8xy^3}{\Delta}$$

CR2: $u_y = -v_x$ $(x, y\neq 0)$

$$\boxed{x=2y}$$

CR1 & CR2 do not have a common solution.

Hence f is not diff'ble when $(x\neq 0, y\neq 0)$.

(b) $$\lim_{\Delta z\to 0}\frac{1}{\Delta z}\big(f(0+\Delta z)-f(0)\big) = \lim \overline{\Delta z}^2/\Delta z^2 = \lim\left(\frac{\overline{\Delta z}}{\Delta z}\right)^2$$ does not exist: $$\lim_{\substack{\Delta z=a+ia\\ a\to 0}} = \left(\frac{1-i}{1+i}\right)^2 \neq \lim_{\substack{\Delta z=a-ia\\ a\to 0}} = \left(\frac{1+i}{1-i}\right)^2$$

Hence f is not diff'ble at $z=0$ either.