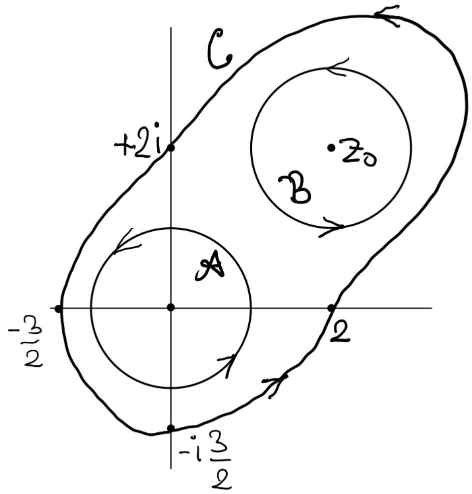
Boğaziçi University
Department of Mathematics
Math 338 Complex Analysis
Spring 2024 – First Midterm

1	2	3	4	5	$\sum$
21 pts	6 pts	3+3+6+6+8 pts	20 pts	27 pts	pts

Date: March 26, 2024 Time: 17:00-19:00	Full Name: A SOLUTION KEY

Throughout the exam $z_0 = 2+2i$ and $\mathcal{A} = \{|z|=1\}, \mathcal{B} = \{|z-z_0|=1\}$ and $\mathcal{C}$ are **positively oriented** curves.



1. Mark the statements below as TRUE or FALSE. No justification is needed in this part. EACH INCORRECT ANSWER CANCELS A CORRECT ONE.

   In this page • $\Omega \subset \mathbb{C}$ is a region; • $\omega \in \Omega$ is a fixed point; • $f : \Omega \to \mathbb{C}$ is a single-valued function; and • $f(z) = u(z) + iv(z)$ with $u, v : \mathbb{R}^2 \to \mathbb{R}$.

1. F f is complex differentiable at ω if and only if f is analytic at ω.
2. T If $f'(w)$ exists then u and v are differentiable at w.
3. F If f is differentiable at w then the partial derivatives u_x,v_x and u_y,v_y exist in a neighborhood of w.
4. T If f is analytic at $w = r_0 e^{i\theta_0}$ then $f'(w) = e^{-i\theta_0}(u_r(w) + iv_r(w))$.
5. T Every polynomial is entire.
6. T $\displaystyle\int_{\mathcal{B}} \frac{1}{z} dz = 0$.
7. F $\displaystyle\int_{\mathcal{A}+\mathcal{B}+\mathcal{C}} \frac{1}{z(z+z_0)} dz = 0$.

2. Write down the Cauchy-Riemann equations in the boxes below.

in terms of u_x, v_x, u_y, v_y: $\boxed{}$

in terms of $u_r, v_r, u_\theta, v_\theta$: $\boxed{}$

3. Define the *complex hyperbolic functions* as follows:

$$\sinh z = \frac{1}{2}(e^z - e^{-z}), \quad \cosh z = \frac{1}{2}(e^z + e^{-z}).$$

(a) Are these functions multi-valued or single-valued? Why?

Single valued since e^z is so.

(b) Are these functions entire? Why?

Entire since e^z is so.

(c) Find the domain of the function $\tanh z \doteq \dfrac{\sinh z}{\cosh z}$.

That is, solve $\cosh z = 0$: $e^z + e^{-z} = e^x \cdot \operatorname{cis} y + e^{-x} \operatorname{cis}(-y) = 0$

$$\Rightarrow \begin{cases} \cos y \cdot (e^x + e^{-x}) = 0 \xRightarrow[e^x > 0, \forall x]{} \cos y = 0 \Rightarrow y = \pi/2 + k\pi,\ k \in \mathbb{Z} \\ \sin y \cdot (e^x - e^{-x}) = 0 \rightsquigarrow \sin y \neq 0 \ \& \ e^x = e^{-x} \Leftrightarrow e^{2x} = 1 \Rightarrow x = 0. \end{cases}$$

Hence $\cosh z = 0 \Leftrightarrow z = i(\pi/2 + k\pi)$

domain = $\boxed{\{ z = i(\pi/2 + k\pi) \mid k \in \mathbb{Z} \}}$

(d) Compute $(\tanh z)'$.

$(\tanh z)' = \boxed{1/\cosh^2 z}$

(e) Compute $\int_{\mathcal{A}+\mathcal{B}-\mathcal{C}} \frac{1}{\cosh^2 z} dz$. Show your work in detail.

On and around $\mathcal{A}, \mathcal{B}, \mathcal{C}$ the integrand has an antiderivative $F(z) = \tanh z$, which is defined everywhere except $i\left(\frac{\pi}{2}+2k\pi\right)$.

So the integral is zero.

result: zero.

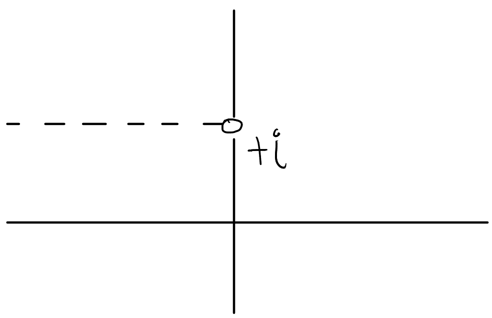
4. Find an open domain $U \subset \mathbb{C}$ as large as possible over which the function $f(z) = \log(z - i)$ is single valued and analytic.

For $z = x + iy$, $f(z) = \log(x + i(y-1))$

$$= \ln r + i\,\mathrm{Arg}(x + i(y-1)) + i2k\pi$$

For single-valuedness, set $k=0$ and $\mathrm{Arg} \in (-\pi, +\pi)$.

This choice corresponds to the domain:



domain = $\boxed{\mathbb{C} - \{x + i \mid x \leqslant 0\}}$

5. Find all the points at which the function $g(z) = e^y e^{ix}$ is analytic. (Here $z = x + iy$.)

$$g(z) = e^y \operatorname{cis} x = \underbrace{e^y \cos x}_{u} + i\,\underbrace{e^y \sin x}_{v}$$

CR eqns: $u_x = -e^y \sin x = e^y \sin x = v_y \overset{e^y \neq 0}{\Rightarrow} \sin x = 0$

$u_y = e^y \cos x = -e^y \cos x = -v_x \Rightarrow \cos x = 0$

CR eqns are never satisfied.

Hence $g(z)$ is nowhere differentiable, nowhere analytic.

domain = $\boxed{\emptyset}$