Boğaziçi University
Department of Mathematics
Math 338 Complex Analysis
Spring 2024 – Second Midterm

1	2	3	4	5	6	$\sum$
15 pts	10 pts	10+12 pts	18 pts	20 pts	8+8 pts	101 pts

Date: April 30, 2024 Time: 13:00-15:00	Full Name: A KEY

You may use every fact that we have already proven in the class. Among those here are two for your convenience:

`Cauchy Integral Formula:` If f is analytic on and inside a positively oriented contour $\mathcal{C}$ and a is a point in the interior of $\mathcal{C}$ then $f^{(n)}(a) = \dfrac{n!}{2\pi i}\displaystyle\int_{\mathcal{C}} \frac{f(z)dz}{(z-a)^{n+1}}$ for every $n \in \mathbb{Z}^{\geq 0}$.

`Extended Liouville Theorem:` If f is entire and if $|f(z)| \leq A + B|z|^k$ for some $k \in \mathbb{Z}^{\geq 0}$; $A, B \in \mathbb{R}^{>0}$ then f is a polynomial of degree at most k.

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1. Mark the statements below as TRUE or FALSE. No justification is needed in this part. EACH INCORRECT ANSWER CANCELS A CORRECT ONE.

1. $\boxed{T}$ If f has antiderivative over a region then f is analytic there. ($f = F'$ on a region. So F is analytic, so is f).
2. $\boxed{T}$ An analytic function is infinitely differentiable.
3. $\boxed{T}$ Given a sequence $(a_n)_{n\in\mathbb{Z}^+}$ of distinct points in $\mathbb{C}$ if there is an entire function g satisfying $g(1/n) = a_n$ for every $n \in \mathbb{Z}^+$, then such a g is unique. (if there were some other h, then $(h-g)(1/n) = 0$ $\forall n$).

2. Let $\mathcal{C}_R$ denote the positively oriented circle centered at 0 with radius R. Write the results (in the form $a + ib$) in the boxes provided. (Each box takes either full or no points.)

$$\int_{\mathcal{C}_{1/2}} \frac{\cosh z\, dz}{(z - i\pi/4)^3} = \boxed{0}$$

analytic inside C_1

$$\int_{\mathcal{C}_3} \frac{\cosh z\, dz}{(z - i\pi/4)^3} = \boxed{i\pi\sqrt{2}/2}$$

$$= \frac{2\pi i}{2!}\cdot \underbrace{(\cosh)''\left(i\frac{\pi}{4}\right)}$$

$$= \cosh(i\pi/4)$$

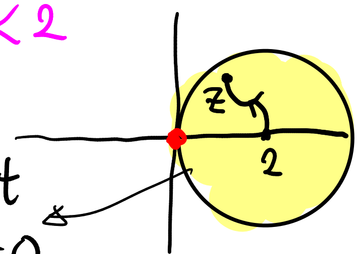
$$= \frac{1}{2}\left(e^{i\pi/4} + e^{-i\pi/4}\right)$$

$$= \cos\pi/4 = \sqrt{2}/2$$

3. (a) Find a power series $\sigma(w) = \sum_{n=0}^{\infty} a_n (w-2)^n$ that equals $f(w) = 1/w$ in a disk neighborhood of $2 \in \mathbb{C}$. What is the largest disk $\Delta \subset \mathbb{C}$ where $\sigma(w) = f(w)$?

$$\frac{1}{w} = \frac{1}{w-2+2} = \frac{1}{2} \cdot \frac{1}{1+\frac{w-2}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{w-2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (w-2)^n$$

when $|w-2| < 2$



this is the largest disk since at $w=0$, f is not defined.

(b) Recall that a contour integral of a power series (in the disk of convergence) can be performed term by term. For any point $z \in \Delta$ and a contour $\mathcal{C} \subset \Delta$ from 2 to z, consider the contour integrals

$$\int_C \sigma(w)dw = \int_C f(w)dw.$$

Evaluating both sides, obtain a power series for Log z around 2 in Δ. (Helping remarks: • Log z is the P.V. of log with branch cut the nonpositive real numbers. • After evaluating each integral above, w should disappear. The results must be a function of z.)

Recall: $\frac{d}{dz} \text{Log} z = \frac{1}{z}$. Therefore for C as above,

$$\int_C f(w)dw = \text{Log}\, z - \text{Log}\, 2$$

Meanwhile, $$\int_C \sigma(w)dw = \sum_{n=0}^{\infty} a_n \int_C (w-2)^n dw = \sum a_n \left.\frac{(w-2)^{n+1}}{n+1}\right|_{w=2}^{w=z}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}(n+1)} (z-2)^{n+1}$$

Hence, $$\text{Log}\, z = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n \cdot n} (z-2)^n$$

4. Consider the function $g(z) = 1/z^2$. Using the result of (3a) above find a power series centered at $z = 2$ and determine its radius of convergence. Explain your work. (Warning: Do not compute the Taylor expansion by explicit computation.)

By (3a), $\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^n$ on Δ. Then

$-\frac{1}{z^2} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{2^{n+1}} (z-2)^{n-1}$. Hence

$\frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (n+1)}{2^{n+2}} (z-2)^n$ on Δ,

since R is preserved under derivation.

5. Suppose f is entire and $|f(z)| \le A + B|z|^{3/2}$. Show that f is a linear polynomial, i.e. its degree is at most 1. (A help: At some point the triangle inequality in the reverse direction might be handy.)

By extended Liouville $f(z) = a + bz + cz^2$, for some $a, b, c \in \mathbb{C}$

Assume $c \neq 0$.

Then $\underbrace{0 \le |c|\cdot|z|^2 - |a| - |b|\cdot|z|}_{\text{for } |c|>0 \,\&\, |z| \text{ large}} \underbrace{\le}_{\text{triangle ineq}} |f(z)| \underbrace{\le}_{\text{given}} A + B|z|^{3/2}$

So for every large $|z|$, $|c|\cdot|z|^2 \le A' + |b|\cdot|z| + B|z|^{3/2}$

which is impossible. Hence c must be 0.

6. (a) Find the Maclaurin series for $\sin z$ by recursively computing the derivatives. Determine the radius of convergence.

Standard computation... Let's do it once more:

$s(z) = \sin z$. $s(0) = 0$, $s'(0) = 1$, $s''(0) = 0$, $s'''(0) = -1$

$s^{(4k)}(0) = 0; \quad {}'' = s^{(4k+1)}(0); \quad {}'' = s^{(4k+2)}(0); \quad {}''\ s^{(4k+3)}(0)$

$$\text{So } \sin z = \sum_{n=0}^{\infty} \frac{s^{(n)}(0)}{n!} z^n = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$$

$R = \infty$, since $\sin z$ is entire.

(b) Find a power series in the form $\sum_{n=0}^{\infty} c_n z^n$ for the function $h(z) = \frac{\sin z}{z}$, $z \neq 0$. Tell very carefully how and why h can be extended to an entire function.

$$h(z) = \frac{\sin z}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k} \quad (z \neq 0)$$

However the RHS is 1 at $z=0$. So the extension $\tilde{h}(z) = \begin{cases} h(z), & z \neq 0 \\ 1, & z = 0 \end{cases}$ is entire, being a power series with $R = \infty$.