Boğaziçi University
Department of Mathematics
Math 234 Advanced Calculus II
Spring 2025 – First Midterm bis

1	2	3	4	$\sum$
PROPOSED		SOLUTIONS		
30 pts	24 pts	24 pts	24 pts	100 pts

Date: May 13th, 2025	Full Name:
Time: 17:00-18:40	

1. TRUE or FALSE. Either prove or refute. Refuting is a proof; you can do this by giving a counterexample.

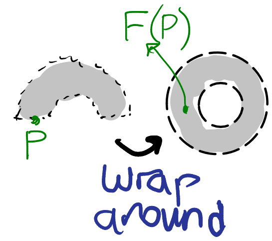
A. Let $g : S \to \mathbb{R}$ be a continuous and bounded function over some bounded Jordan measurable set $S \subset \mathbb{R}^2$. Then there is some $x_0 \in S$ satisfying

$$g(x_0) \cdot \text{area}(S) = \iint_S g(x)dA.$$

FALSE. For connected S, this would be the Mean Value Thm for integrals. Produce a counterexample where S is disconnected.

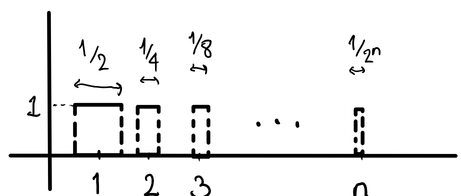
B. Let $\emptyset \neq V \subset U \subset \mathbb{R}^2$ be open sets with $\text{cl}(V) \subset U$, $F : U \to \mathbb{R}^2$ be a C^1 function, $DF(\mathbf{x})$ be nonsingular for all $\mathbf{x} \in V$. Then $F(\partial V) = \partial F(V)$.

FALSE. $U = \mathbb{R}^2 - \{0\} \supset V = \left\{ \begin{array}{c} 1 < x^2+y^2 < 4 \\ 0 < \theta < \pi \end{array} \right\}$, $F : (x,y) \longmapsto \underbrace{(r\cos 4\theta, r\sin 4\theta)}_{\text{Better, write this in } x \,\&\, y}$



C. If $\int_0^\infty f(x)dx$ is convergent then $f(x)$ goes to 0 as $x \to \infty$.

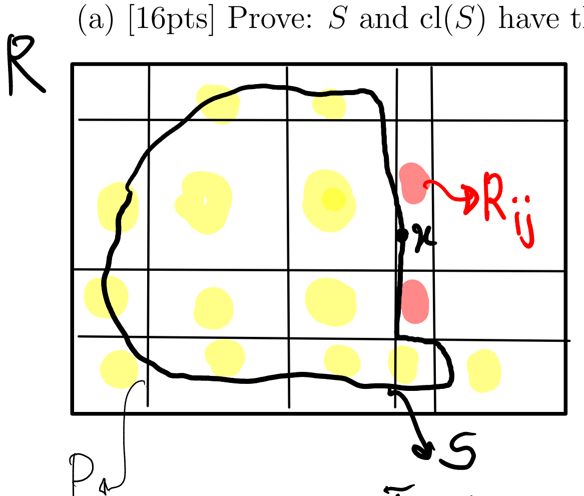
FALSE. $f : [0,\infty) \to \mathbb{R}$



$$\int_0^\infty f = \sum_{n=1}^\infty \frac{1}{2^n} < +\infty \text{ but } f \not\to 0$$

2. Consider a subset S of $\mathbb{R}^2$ and a bounding rectangle $R \supset S$. For any partition P of R the upper sum of the characteristic function χ_S of S is defined, as you know. Recall that the outer area of S is the infimum of all such upper sums over all possible partitions.

(a) [16pts] Prove: S and $\text{cl}(S)$ have the same outer area.



In the figure, the yellow rectangles of P touch S, and the red rectangles touch ∂S (and not S).

If $R_{ij} \cap \partial S \neq \emptyset$ then these are the two possible cases. In yellow case $|R_{ij}|$ counts in the upper sum of both S & $\text{cl}(S)$.

In the red case $\exists x \in \partial S$, $x \notin S$, $x \in \partial R_{ij}$.

Here construct a refinement P' of P s.t. such x's are covered by rectangles of total area $< \varepsilon$ (can be done since ∂R_{ij} has 0 measure). For this P', the upper sums differ by $< \varepsilon$. $\varepsilon > 0$ is arbitrary. So we're done.

(b) [8pts] Does this mean S is Jordan measurable (i.e. ∂S has zero content)? Why? Why not?

- When S is nonempty open in (a), $cl(S) - S = \partial S$
- It's known S is J. measurable $\Rightarrow$ (outer − inner) can be made arbitrarily small.
- But the converse need not be true even if S is open.
- $\mathbb{Q}^2 \cap [0,1]^2 \subset \mathbb{R}^2$ has $\partial = [0,1]^2$, which has not zero J. measure.

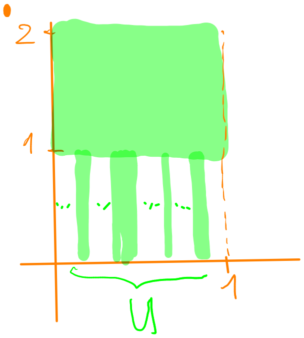
this is bonus

- Here is an OPEN, connected example in $\mathbb{R}^2$:

open $U = \bigcup_{j=1}^{\infty} (a_j, b_j)$, $(a_j, b_j) \ni r_j$ → j-th rational number in $(0,1)$

with $|b_j - a_j| < 1/2^{j+1}$

claims
- $\partial U = [0,1] - U$
- ∂U does not have zero J. measure.



open, with its boundary not content zero