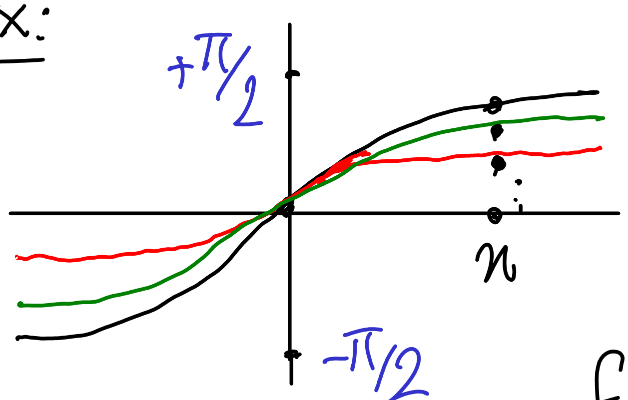
SEQUENCES & SERIES OF FUNCTIONS

§ Sequences of functions.

$(f_n)_{n\geq 1}^{\infty}$ a sequence of fn-s $f_n : S \to \mathbb{R}$, $S \subset \mathbb{R}$

ex:



$f_1(x) = \arctan x$

$f_k(x) = \frac{1}{k}\arctan(kx)$

$f_k(0) = 0, \forall k.$

"In the limit" we have the constant fnc 0.

defn (pointwise convergence) We say (f_k) converges to f pointwise if for each x,

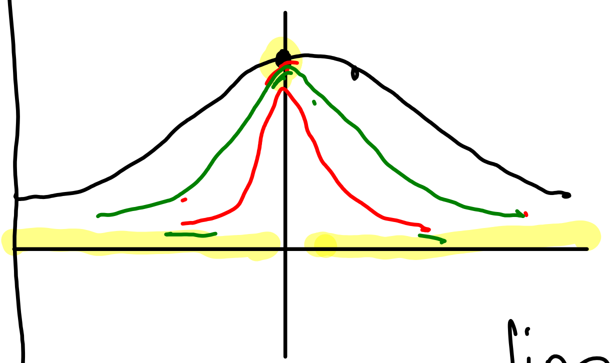
$$\lim_{k\to\infty} f_k(x) = f(x).$$

f is called the pointwise limit of (f_k).

back to ex. $f_k(x) \longrightarrow 0$ pointwise

because $\left|\frac{1}{k}\arctan kx - 0\right| \leq \frac{\pi/2}{k} \xrightarrow{k\to\infty} 0$.

ex. $g_k(x) = \dfrac{1}{k^2x^2+1}$. Observe $f_k'(x) = g_k(x)$.



$\forall k,\ g_k(0) = 1$

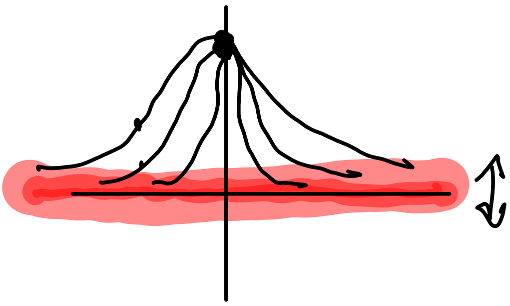
$$\lim_{k\to\infty} g_k(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases} =: g(x)$$

Observe.

- All f_k's are cont & f is cont too.
- All g_k's are cont but g is not.
- All f_k''s are cont but the pointwise limit is not diff'ble.

conclusion. Pointwise convergence is weak if we're interested in preserving continuity, derivatives, integration etc.

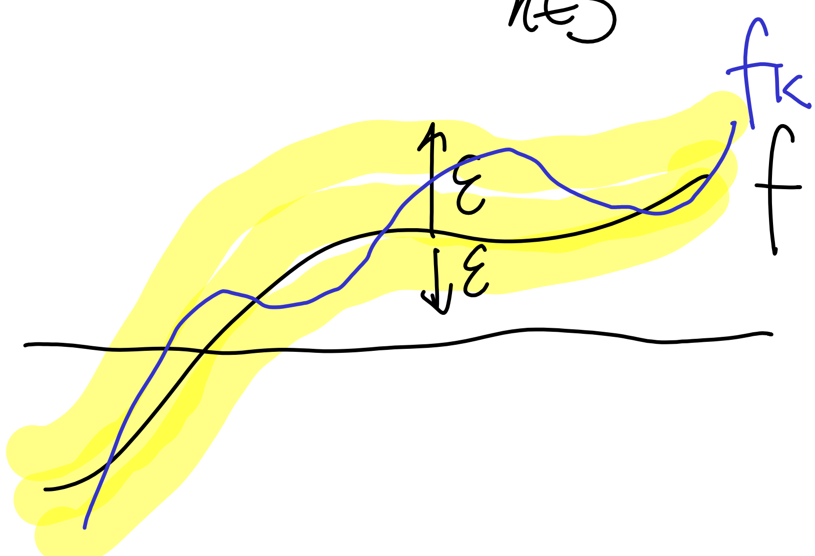
§ Uniform convergence



<u>defn</u>: (f_k) on $S\subset\mathbb{R}$ is said to <u>converge to</u> $f:S\to\mathbb{R}$ <u>uniformly</u> if given $\varepsilon>0$ there is some $K\in\mathbb{Z}^+$ s.t.

$$n>K \Rightarrow |f_n(x)-f(x)|<\varepsilon \ \forall x\in S.$$

i.e. for $n>K$, $\sup\limits_{x\in S}|f_n(x)-f(x)|\leq\varepsilon$.

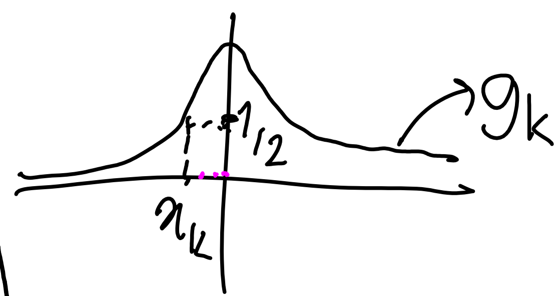


That is, for large index, the graph f_k lies in a tube around the graph of f with radius ε.

<u>prop</u>. $f_k\to f$ uniformly $\iff$ $\exists$ sequence of positive fixed #s (C_k) s.t. $\forall x\in S$, $|f_k(x)-f(x)|\leq C_k$ & $C_k\to 0$. exercise!

thm 1. Suppose f_k's are cont & $f_k \to f$ uniformly. Then f is cont too.

back to ex. $g_k \longrightarrow g$ pointwise but not uniformly.



$\forall k, \exists x_k$ with $g_k(x_k) = \frac{1}{2}$.

None of g_k can be close to g, closer than given $\varepsilon < \frac{1}{2}$.

§ Series of fncs.

$(f_k(x))_{k=1}^{\infty}$, $f_k : S \to \mathbb{R}$. Consider $\sum_{k=1}^{\infty} f_k(x)$.

Set $s_n(x) = \sum_{k=1}^{n} f_k(x)$. Let $s_n(x) \longrightarrow s(x)$ pointwise.

defn: For every $x \in S$ where $s(x)$ exists we say that $\sum f_k(x)$ converges to $s(x)$ pointwise. If $S_n \longrightarrow S$ uniformly over S we say that $\sum f_k(x)$ converges to $s(x)$ uniformly over S.

ex. $\sum_{n=1}^{\infty} x^n$ is a series of fncs.

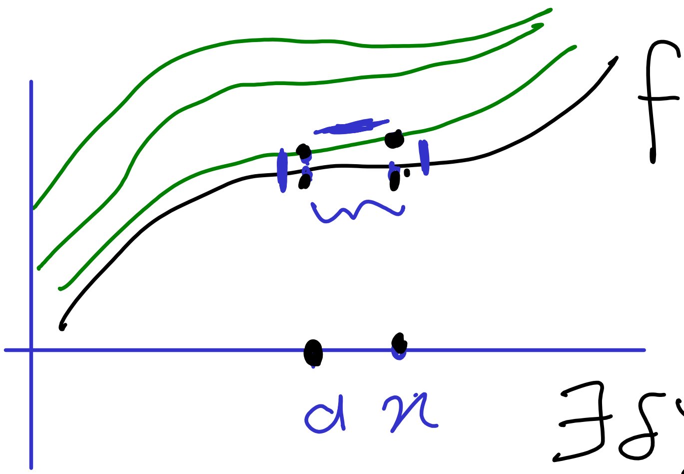
- $x=0$: $\sum 0 = 0$.
- $x=1$: $\sum 1 \longrightarrow \infty$.
- $x=-1$: $\sum (-1)^n$ diverges.
- $|x|>1$: $\sum x^n$ diverges. $\left(x^n \not\to 0 \right)$
- $-1 < x = r < 1$: $\sum_{n=0}^{\infty} r^n = \lim_{k\to\infty} \underbrace{\frac{1-r^{k+1}}{1-r}}_{S_k(r)} = \frac{1}{1-r}$. $S_k(x)$ converges absolutely.

If $|x|<1$ the $S_k(x) = \frac{1-x^{k+1}}{1-x} \xrightarrow[s(x)]{\text{pointwise}} \frac{1}{1-x}$

thm 2. If f_k's are cont & $\sum f_k(x)$ converges uniformly then the sum of this series is cont.

Pf: $S_k(x)$ are cont. Now use thm 1.

<u>proof of thm 1.</u>



We know **a)** given $\varepsilon > 0$ $\exists N$ s.t. if $k > N$ $\forall x$, $|f_k(x) - f(x)| < \varepsilon$.

b) All f_k's are cont: $\exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |f_k(x) - f_k(a)| < \varepsilon/3$

Fix $a \in S$. We'll prove f is cont at a.

Given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-a| < \delta$

By (a), for large k, $|f_k(x) - f(x)| < \varepsilon/3$
& $|f_k(a) - f(a)| < \varepsilon/3$.

$$|f(x)-f(a)| \leq |f(x)-f_k(x)| + |f_k(x)-f_k(a)| + |f_k(a)-f(a)| < \varepsilon.$$

∎

Sequences of fncs | Series of fncs

uniform convergence

- $\forall \varepsilon > 0\ \exists N$: $k > N \Rightarrow |f_k(x) - f(x)| < \varepsilon,\ \forall x \in S$

$\Leftrightarrow$ $\sup_{x \in S} |f_k(x) - f(x)| \leq \varepsilon$

$\Leftrightarrow$ $\forall x \in S,\ |f_k(x) - f(x)| < C_k$ & $C_k \xrightarrow{k\to\infty} 0$

$\sum_{n=1}^{\infty} f_n$ is unif. conv. $\Leftrightarrow$ $S_k = \sum_{n=1}^{k} f_n(x)$ converges uniformly

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thm 1

f_k's cont, unif conv to f. Then f is cont.

$$\lim_{x\to a} \lim_{k\to\infty} f_k(x) = \lim_{x\to a} f(x) = f(a)$$

$$\lim_{k\to\infty} \lim_{x\to a} f_k(x) = \lim_{k\to\infty} f_k(a)$$

f_n's cont and $\sum_{n=1}^{\infty} f_n$ unif conv to s then $s(x)$ is cont

$$\lim_{x\to a} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x\to a} f_n(x)$$

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thm 3

$f_k \to f$ uniformly & all intgble then $\int f = \lim_{k\to\infty} \int f_k$

$\sum f_k \to s$ uniformly & all intgble then $\sum \int = \int \sum$.

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thm 4

f_k C^1; $f_k \to f$ pointwise; $f_k' \to g$ unif. Then

$$g(x) = \lim_{k\to\infty} \frac{d}{dx} f_k = \frac{d}{dx} \lim_{k\to\infty} f_k = f'(x)$$

f_k C^1; $\sum f_k \to s(x)$ pointwise, $\sum f_k'$ converges unif. Then

$$\frac{d}{dx} s(x) = \sum \frac{d}{dx} f_k$$

§ Weierstrass M-test.

$(f_n)_{n=1}^{\infty}$ on $S \to \mathbb{R}$. Suppose $\exists\, M_n \in \mathbb{R}$

(i) $|f_n(x)| < M_n \ \forall x \in S$; (ii) $\sum M_n$ is convergent.

Then $\sum f_n$ is absolutely convergent for $\forall x \in S$

& $\sum f_n$ is uniformly convergent over S.

<u>proof</u>: By comparison, (i) & (ii) $\Rightarrow$ the 1st claim.

Since $\sum f_n(x)$ is abs. convgnt $\forall x$ then

$\sum f_n(x)$ is convergent for $\forall x$, say to $s(x)$.

$$\forall x, \ |s(x) - s_k(x)| = \left|\sum_{n=k+1}^{\infty} f_n(x)\right| \underset{\text{(i) \& tri. ineq.}}{<} \underbrace{\sum_{n=k+1}^{\infty} M_n}_{C_k \to 0} \xrightarrow[k\to\infty]{} 0 \quad \text{by (ii)}$$

<u>ex.</u> • $\sum_{n=0}^{\infty} x^n$ on $[-r, r]$ with $r < 1$.

Since $|x^n| < r^n$ & $\sum r^n$ is convergent

by Weierstrass M-test $\sum x^n$ is unif convgnt on $[-r, r]$.

& abs. convgnt

Moreover: $\sum_{n=0}^{\infty} x^n$ is unif convgnt on $(-1, 1)$.

& abs convgnt

- The Taylor expansion of $\log(1+x)$ around $x=0$:

$$\underbrace{\sum_{1}^{\infty} (-1)^n \frac{x^n}{n}}_{f_n(x)} \qquad |f_n(x)| \leq \frac{r^n}{n} \text{ over } [-r,+r], \quad r<1.$$

$$\& \sum_{n=1}^{\infty} \frac{r^n}{n} \text{ is convergent } \text{(by ratio test)}$$

By M-test, the series is unif (& abs) convergent over $[-r,+r]$ $\forall r<1$.

§ Integration & derivation

<u>thm 3</u>: Let $f_k \longrightarrow f$ uniformly & f_k, f be integrable over a measurable $S \subset \mathbb{R}^m$. Then

$$\lim_{k\to\infty} \int_S f_k = \int_S \lim_{k\to\infty} f_k = \int_S f.$$

<u>proof</u>. $$\left|\int_S f_k - \int_S f\right| \leq \int_S |f_k - f| \leq \int_S C_k = C_k \int_S dx$$

$\& \ C_k \to 0$

$$= C_k \cdot \mathrm{vol}(S) \xrightarrow[k\to\infty]{} 0$$

coral. If $\sum f_k(x) \longrightarrow s(x)$ unif. & f_k's & s are integrable then

$$\sum \int f_k = \int \sum f_k = \int s.$$

thm4. Assume f_k is C^1 and $f_k \to f$ pointwise and $f_k' \longrightarrow g$ uniformly on $[a,b]$.

Then $g(x) = \lim\limits_{k\to\infty} f_k' = \frac{d}{dx} \lim\limits_{k\to\infty} f_k = f'(x)$

proof. f_k' are cont. By thm 1, g is cont.

$$\int_a^x g(t)\,dt = \int_a^x \lim_{k\to\infty} f_k' \underset{\text{thm 3}}{=} \lim_{k\to\infty} \int_a^x f_k'$$

$$\overset{\text{FTC}}{=} \lim_{k\to\infty} \left(f_k(x) - f_k(a) \right) = f(x) - f(a)$$

Hence $f(x) = f(a) + \int_a^x g(t)\,dt$.

Check: $f'(x) = \underset{\text{F.T.C.}}{0} + g(x)$.

Corol. Suppose $\sum f_k(x)$ are convergent $\forall x \in S$ where f_k's are C^1; $\sum f_k'(x)$ converges uniformly to $s(x)$.

Then $\sum f_k$ is C^1 and $\frac{d}{dx}\sum f_k(x) = s(x)$.

ex: $\sum_{n=1}^{\infty} \frac{x^n}{n}$ with sum $s(x)$.

$$s'(x) = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n \overset{|x|<1}{=} \frac{1}{1-x} \Rightarrow s(x) = -\ln(1-x)$$

term-by-term differentiation

thx to ①: • x^n/n is C^1; • $|x|<1$ for converges pointwise • $\sum_0^\infty x^n$ unif convgs. by M-test.

§ Power series

For power series you don't have such "ugly" behavior. $\sum f_n$ where $f_n = a_n(x-c)^n$

lemma. If $\sum_0^\infty a_n x^n$ is convergent for $x = x_0$ then it's absolutely convergent for all $|x| < x_0$.

proof. $a_n x_0^n \longrightarrow 0 \Rightarrow |a_n x_0^n| < K$

$$|a_n x^n| = |a_n| \cdot \left|\left(\frac{x}{x_0}\right)^n\right| \cdot |x_0^n| < K \cdot \left|\frac{x^n}{x_0^n}\right|$$

By comparison, $\sum |a_n x^n|$ is convergent for $\left|\frac{x}{x_0}\right| < 1$.

thm 1: Every power series that converges at 0 has a radius of convergence $R \in [0, +\infty) \cup \{+\infty\}$, i.e

$\forall |x| < R$, $\sum a_n x^n$ is abs. conv.

$\forall |x| > R$, $\sum a_n x^n$ is divergent.

proof: $R = \sup\{x_0 : \sum a_n x_0^n \text{ is convergent}\}$

thm 2: $\forall r<R$, $\sum a_n x^n$ converges uniformly on $[-r,r]$. Hence its sum is cont on $(-R,R)$.

proof. $|a_n x^n| < |a_n| r^n =: C_n$, $\sum C_n$ convgs.

by M-test $\sum a_n x^n$ convgs unifly.

thm 3: term-by-term integration

ex: $\displaystyle\int_0^x \frac{\sin t}{t}\,dt = \int_0^x \sum_0^\infty \frac{(-1)^n \cdot t^{2n+\cancel{1}}}{(2n+1)!}$

thm $\rightarrow$ $\displaystyle = \sum_0^\infty \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)\cdot(2n+1)!}$

thm 4: $\sum_0^\infty a_n x^n$ has radius of convergence R is equal to that of $\sum_1^\infty n a_n x^{n-1}$ (R').

proof. $\forall |x|<R'$: $\sum n a_n x^{n-1}$ is abs convergent &

$$|a_n x^n| = \frac{|x|}{n} |n a_n x^{n-1}| \underset{\text{large } n}{\leq} |n a_n x^{n-1}|, \forall n \text{ large}.$$

Hence $|x|<R$. Then $R' \leq R$.

Conversely, $|x| < r < R$,

$$|na_n x^{n-1}| = \underbrace{\frac{1}{|x|}\left|\boxed{n}\,\frac{x^n}{r^n}\right|}_{\to 0 \text{ as } n\to\infty} \cdot r^n |a_n| \leq |a_n r^n|$$

$\hookrightarrow$ for large n

Hence $|x| < R'$ and $R \leq R'$.

thm5: term-by-term differentiation

Let $\sum a_n x^n$ have radius of convergence $R > 0$.
By thm 4, $\sum n a_n x^{n-1}$ has rad of conv R.
By thm 2, $\sum n a_n x^{n-1}$ convgs unif. in $(-R, +R)$.
By prev thms, $\left(\sum a_n x^n\right)' = \sum n a_n x^{n-1}$ over $(-R, +R)$.
Moreover $\sum a_n x^n$ is C^∞!

corol. Let $f(x) = \sum_0^\infty a_n x^n$ with $R > 0$. Then the Taylor series of f is $\xrightarrow{\text{at } x=0}$ the given power series.

proof. $f^{(n)}(0) \underset{\text{by thm 5}}{=} n!\,a_n + \cdots x + \cdots x^2 \Big|_{x=0} = n!\,a_n$.

Taylor series at $x=0$:

$$\sum_0^\infty \frac{f^{(n)}(0)}{n!} x^n = \sum_0^\infty a_n x^n.$$

corol. If $\sum a_n x^n = \sum b_n x^n$ with rad of conv R

then $\forall n$, $a_n = b_n$.

proof. Both are the Taylor series of the same fnc.

Summary.

Let $\sum a_n x^n$ converge at $x=R$. Then

- $\sum a_n x^n$ converges absolutely for $|x|<R$
- $\sum a_n x^n$ converges uniformly on $[-r,r]$, $\forall r<R$.
- The sum is continuous on $(-R,+R)$.
- The sum is C^∞ on $(-R,+R)$.

(proof thru the fact that R is preserved after derivation.

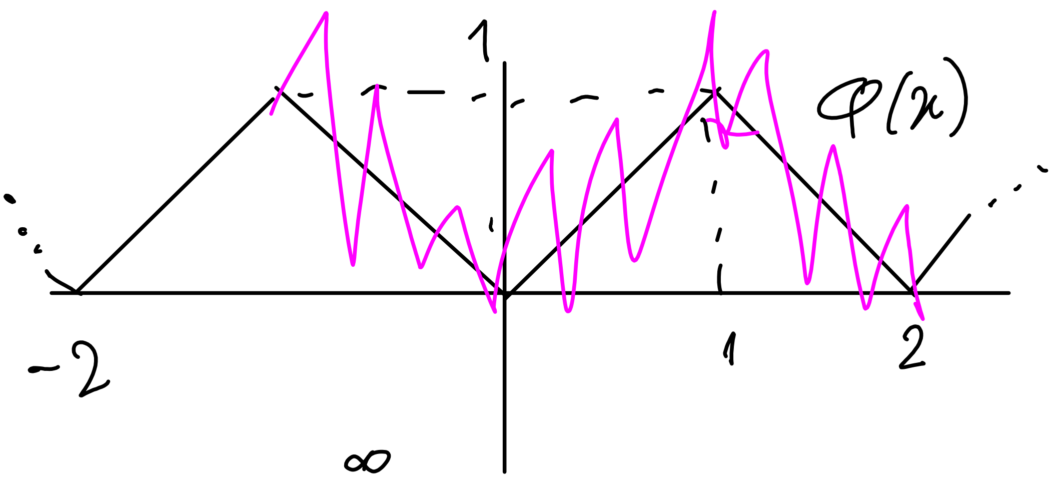
Also: ☆ If $\sum a_n x^n$ is abs convergent on at $x=\pm R$ then, setting $M_n=|a_n|\cdot R^n$, $\sum a_n x^n$ is abs & unif convergent on $[-R,R]$.

Lastly work out the proof of Abel's Theorem. If $\sum a_n x^n$ is convergent at $x=R$ then $\sum a_n x^n$ is unif. convergent over $[0,R]$. Therefore the sum cont at R too.

BONUS:

∃ A continuous function which is nowhere diff'ble.

(John McCarthy, Monthly AMS, Dec 1953)

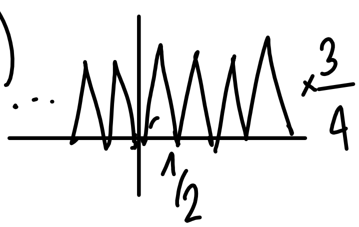


$$f(x) = \sum_{n=0}^{\infty} \underbrace{\left(\frac{3}{4}\right)^n \varphi(4^n x)}_{f_n(x)} \quad \text{is such a fnc.}$$

$$S_0(x) = \varphi(x)$$

$$S_1(x) = \varphi(x) + \frac{3}{4}\varphi(4x)$$

$\varphi(4x)$



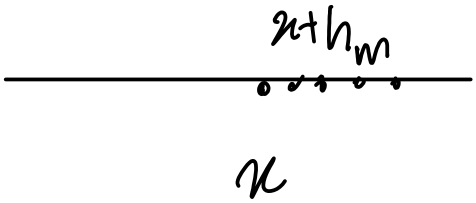
facts.

① f is cont: $\begin{cases} \text{Each } f_n \text{ is cont.} \\ |f_n(x)| \leq \left(\frac{3}{4}\right)^n = M_n \ \& \ \sum M_n \text{ convgnt.} \end{cases}$

By Weierstrass M-test, we've unif convgnce.
Hence f is cont.

(2) $\forall x$, f is not diff'ble at x. We show

$$\lim_{h\to 0} \frac{1}{h}\left[f(x+h)-f(x)\right]$$ does not exist.

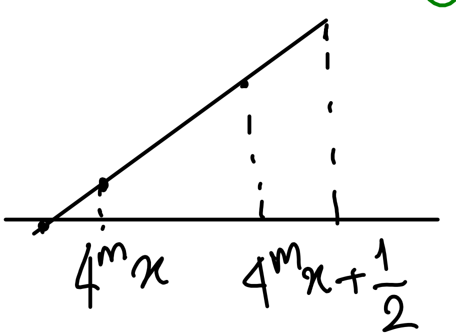


Equivalently, we construct $(h_m)_{m=0}^{\infty}$, $h_m \longrightarrow 0$ such that

$$\underbrace{\lim_{m\to\infty} \frac{1}{h_m}\left[f(x+h_m)-f(x)\right]}_{(*)}$$ does not exist.

(3) $h_m \doteq \pm\frac{1}{2}4^{-m} \longrightarrow 0$

↳ choose $+$ or $-$ s.t. there is no integer between $4^m x$ & $4^m x \pm \frac{1}{2}$.



4 $(*)$: $\dfrac{1}{h_m}\left[\sum_{n=0}^{\infty} f_n(x+h_m) - \sum_{n=0}^{\infty} f_n(x)\right]$

calculate

$$\Delta_{m,n} = \frac{1}{h_m}\left[f_n(x+h_m) - f_n(x)\right]$$

(a) $n>m$: $\Delta_{m,n} = 0$

(b) $n=m$: $\Delta_{m,m} = 3^m$

(c) $n<m$: $|\Delta_{m,n}| \leqslant 3^n$

prove these

$$(*) = \left|\frac{1}{h_m}\sum_{n=0}^{m} f_n(x+h_m) - f_n(x)\right| = \left|\sum_{n=0}^{m}\Delta_{m,n}\right|$$

$$\geqslant \Delta_{m,m} - \sum_{n=0}^{m-1}|\Delta_{m,n}|$$

$$\geqslant 3^m - \frac{1-3^m}{1-3} = \frac{1}{2}(3^m+1) \xrightarrow[m\to\infty]{} +\infty$$

$$\Delta_{m,n} = \frac{1}{h_m}\left[\left(\frac{3}{4}\right)^n \varphi\left(4^n(x+h_m)\right) - \left(\frac{3}{4}\right)^n \varphi(4^n x)\right]$$

$$= \pm 2 \cdot 4^{m-n} \cdot 3^n \left[\varphi\left(4^n x \pm \underbrace{\tfrac{1}{2} 4^{n-m}}\right) - \varphi(4^n x)\right]$$

(a) $n > m$: even $\Rightarrow$ $= \varphi(4^n x) - \varphi(4^n x) = 0$

(b) $n = m$: $\Delta_{m,m} = \pm 2 \cdot 3^m \cdot \underbrace{\left[\varphi\left(4^m x \pm \tfrac{1}{2}\right) - \varphi(4^m x)\right]}_{= 4^m x \pm \frac{1}{2} - 4^m x = \pm \frac{1}{2}}$

$= 3^m$

(c) $n < m$: $|\Delta_{m,n}| \leq 2 \cdot 4^{m-n} \cdot 3^n \cdot \frac{1}{2} 4^{n-m} = 3^n$